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Systematic Statistics Used for Data Compression in Space Telemetry

Isidore Eisenberger

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CALIFORNIA INSTITUTE OF TECHNOLOGY

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CONTENTS

TABLES

TABLES (Cont'd)

FIGURES

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ABSTRACT

13279

The need for data compression, a consequence of the demands made on the telemetry system of a space vehicle, prompts consideration of the use of sample quantiles in estimating population parameters and obtaining tests of goodness of fit for large samples. In this paper optimal unbiased estimators of the mean and standard deviation are given using up to twenty quantiles when the population is normal. Moreover, the estimators are relatively insensitive to deviations from normality. A distribution-free goodness-of-fit test is presented based on the sum of the squares of four quantiles after an orthogonal transformation to independent normal deviates. If a frequency function is of the form $f(x;p) = pf_1(x) + (1-p)f_2(x), 0 < p < 1$, where f_1 and f_2 are normal frequency functions, the distribution is likely to be bimodal. Another goodness-of-fit test is obtained using four quantiles, which is likely to have considerable power with a null hypothesis of normality and the alternative hypothesis of bimodality. The "data compression ratios" obtained with the use of a quantile system can be on the order of 100 to 1.

AUTHOR

I. INTRODUCTION

This paper introduces to statisticians a new area of application of statistics to the space program. Each deep space probe or planetary probe represents a great expenditure of effort, and as much usable information must be obtained from each shot as possible. Arbitrarily sophisticated data processing equipment is available here *on Earth*; the restriction comes rather in the *space communication link* between the probe and Earth. Previous work on inefficient statistics (Ref. 1-4) has centered on the problem of simple processing of large amounts of data. The problem in data compression is to precondition the data aboard the spacecraft so that useful information can be obtained from the received data. There are many different data compression schemes under consideration suitable for various classes of experiments; it is the purpose of this paper to illustrate one of these methods, showing the complexity of the compression equipment, the savings in amount of data which has to be transmitted, and the uses to which the received data can be put.

We shall consider, as defined in Ref. 1, certain functions of order statistics called *systematic statistics*. First, let us recall the definition of the pth *quantile* z_p of a (cumulative) distribution function $F(x)$ for $0 < p < 1$. This z_p is defined as the lower limit of all μ such that $F(\mu) > p$. For $p = 1/2$, z_p is called the *median*; for $p = 1/4$, the *first quartile*, etc.

We shall be forming z_p from the cumulative sample distribution, and any statistic based on these quantiles (or order statistics) is called a *systematic statistic* by Mosteller (Ref. 1). The point is that quantiles can be computed aboard a space probe very easily and then transmitted to Earth. But let us consider what "computed very easily" means. An on-board computer is needed which does not use any *arithmetic* operations, since to do these operations requires complex arithmetic units and stored programs. Instead, only counters and accumulators should be used. This is basically the same reason for which Mosteller studied order statistics – ease of sorting and computing quantiles with the then available punched card equipment.

The main use envisioned at present for systematic statistics is in particle count experiments, but most space experiments to date have been of this nature. A so-called functional diagram of a particle counter is shown in Fig. 1. Here four quantiles are being used: 0.067, 0.291, 0.709, 0.933, corresponding to an optimal choice of four quantiles given later.

The system may be described as follows: The 6-bit *input counter* counts the number of incoming particles per second. On command from a *control unit* (not shown), the *distributor* at the end of each second puts a 1 in the *storage register* indexed by the number of particles counted. (The experiment is presumably arranged so that counts of more than 63 in 1 sec are unlikely.) The control unit also sets the input counter to 0 at the end of each second. Once every 1,000 sec the control unit causes the contents of storage registers $R_0, R_1, \cdots, R_{63}$ to be added sequentially into the *accumulator*, which has been set to 0 at the end of the previous quantile computation; after being loaded into the accumulator, the registers are set to 0, to await the next thousand-second cycle. After the contents of each R_i are added into the accumulator, each of the four *comparators* compares the number in the accumulator with, for example, 67, 291, 709, 933 = q_1, q_2, q_3, q_4, respectively. If the sum in the accumulator is less than q_1, the accumulator keeps accumulating. As soon as the accumulator sum is at least q_1, the number in the comparator, C_1, which also equals the number in the accumulator, is transferred with erasure via the *transfer* to the proper *quantile register*, which is in addition nondestructively loaded via the transfer with the index of the register counter last emptied into the accumulator. The register counter then adds the next R_i's into the accumulator until the first time that q_2 is equalled or exceeded. The quantile computation is completed when the fourth quantile has been computed, although the accumulator is still being loaded for the remaining number of seconds until the thousandth. (The extra information as to what fraction the jth quantile *actually* was is useful; thus one may want to know that the first time the number in the accumulator exceeded 67 it was actually 81.) The control unit then empties the four quantile registers into the telemetry channel and gives the pulse at second 1,001 which empties the input counter into the distributor for the start of the next run of 1,000; this assumes that that quantile computation described above takes place in less than 1 sec, a reasonable assumption.

This computing equipment is much simpler than an arithmetic unit which computes, for example, means and variances. Furthermore, the assumptions discussed below necessary to *compute* (on Earth) the mean and variance

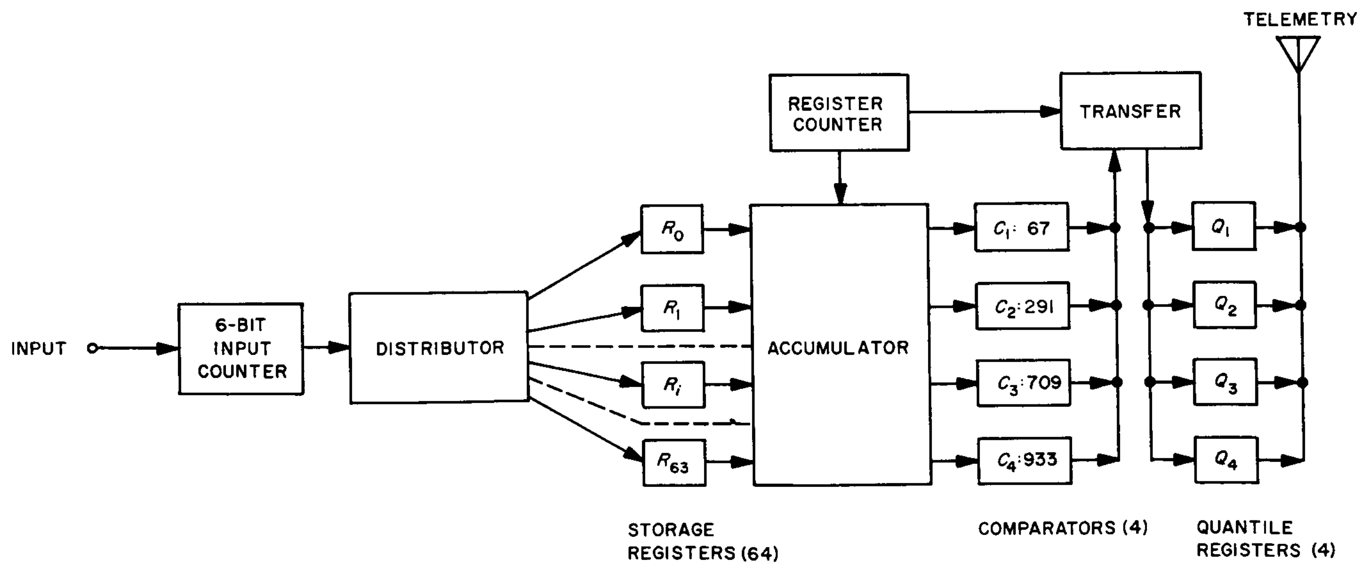

Fig. 1. Data compression system using quantiles

from these quantiles are not sensitive to departures from normality (or actually from the *Poisson* distribution). The data compression ratio, that is, the ratio of transmitted bits, or zeros and ones in a "raw data" system, to the number of bits necessary in this quantile system, may be computed as follows. (Of course, there still remains the question of what can be done with the received quantiles.) The 4 quantiles take 6 bits each, since each quantile is an integer between 0 and $63 = 2^6 - 1$. The excess of the number actually in the accumulator when the quantile was equalled or exceeded and the value of each q_i will take not more than, say, 6 bits, which assumes in effect that no R_i has more than 63 in it, that is, that no number of counts per sec occurs more than 63 times in 1000 sec. The total number of bits required every 1000 sec is $(6 + 6) \times 4 = 48$, since there are 4 quantiles. The raw data system with 6-bit words would take $1000 \times 6 = 6000$ bits every 1000 sec. The data compression ratio is thus $6000/48 = 125$. But it must be seen how *useful* the resulting compressed data is. First the use of quantiles for estimation will be considered; then an application will be given to tests of hypotheses.

II. UNBIASED ESTIMATES OF THE MEAN, μ, AND THE STANDARD DEVIATION, σ, OF A NORMAL POPULATION

Consider a sample of n values, where n is, say, greater than 200, drawn from an (approximately) normal population with a continuous cumulative distribution function $F(x)$ and probability density function $f(x) = F'(x)$. Following Cramér (Ref. 5), let ζ_p denote the pth quantile or, alternatively, the quantile of order p of the distribution F, i.e., $F(\zeta_p) = p$, and let z_p denote the corresponding quantile of the *sample* cumulative distribution.

Cramér shows that the joint distribution and the marginal distributions of two (and indeed of any number of) sample quantiles z_{p_1} and z_{p_2} are asymptotically normal as $n \to \infty$. The means of the limiting distributions are the corresponding quantiles ζ_{p_1} and ζ_{p_2} of the population and the asymptotic second central moments and correlation are given by

$$\sigma_1^2 = \frac{p_1 q_1}{n f_1^2}$$

$$\sigma_2^2 = \frac{p_2 q_2}{n f_2^2}$$

$$\rho_{12} = \left(\frac{p_1 q_2}{p_2 q_1}\right)^{1/2}$$

where $f_m = f(\zeta_m)$, $p_1 < p_2$ and $q_i = 1 - p_i$, $i = 1, 2$.

Asymptotically unbiased estimators of μ and σ will be constructed from linear combinations of k quantiles, for $k = 1,2,3,4,6,8,10 \cdots 20$, suitably chosen to minimize the asymptotic variance of the estimates. With no loss in generality it may be assumed that the underlying normal population has zero mean and unit variance. It is useful to define a measure of the efficiency of estimates obtained from quantiles. Since the sample mean

$$\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$$

is an efficient statistic for estimating μ, and

$$S = \left(\frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2 \right)^{1/2}$$

is an asymptotically unbiased statistic for estimating σ, comparison will be made with these statistics. Thus if $\hat{\mu}$ and $\hat{\sigma}$ are estimates of μ and σ obtained from quantiles, the efficiencies of $\hat{\mu}$ and $\hat{\sigma}$ are defined as

$$\text{eff}(\hat{\mu}) = \frac{\sigma^2}{n\,\sigma^2(\hat{\mu})}$$

$$\text{eff}(\hat{\sigma}) = \frac{\sigma^2}{2n\,\sigma^2(\hat{\sigma})}$$

To apply the results obtained here to statistical experiments performed aboard a space probe, the order of the quantiles must be specified in advance; the *same set* of quantiles must be used in estimating both μ and σ as well as in the goodness-of-fit tests to be subsequently described. Nevertheless, in the interest of greater generality, optimal estimators for μ and σ will be constructed independently. In addition, *suboptimum* sets of quantiles will be given, resulting from minimizing the linear combination $\sigma^2(\hat{\mu}) + b\sigma^2(\hat{\sigma})$ for $b = 1,2$ when $k = 4$ and for $b = 1,2,3$ when $k > 4$. Estimators for μ and σ will be constructed in each case, both using the same set of quantiles.

The estimation of the mean and standard deviation of a normal distribution from quantiles has been considered previously by others, among whom are K. Pearson (Ref. 6), F. Benson (Ref. 4), H. P. Stout and F. Stern (Ref. 3), and, notably, by F. Mosteller (Ref. 1) and J. Ogawa (Ref. 2, pp. 47–55, 272–283). Mosteller gives estimators for the mean using one, two and three quantiles and estimators for the standard deviation using two, four and eight quantiles. The present method of constructing the estimators departs from his essentially in that *weighted* rather than simple averages are taken when the number of quantiles exceeds two, resulting in greater efficiencies. Ogawa (pp. 47–55) defined the relative efficiency of his estimators as the ratio of the amount of information in Fisher's sense derived from the joint distribution of the sample quantiles to that derived from the original whole sample. In the case of a normal population, this definition agrees exactly with the definition of efficiency given here for both the mean and standard deviation. By applying the (Gauss-Markoff) theorem of least squares, Ogawa derived the best linear unbiased estimators for a fixed set of k quantiles, and, by maximizing the relative efficiencies of the estimators, he determined the optimal spacing to be used in choosing the quantiles. The present method of arriving at the optimum spacing consists of determining the set of quantiles which minimizes the variance of the estimate. Although the method given here for finding the best linear unbiased estimators differs from Ogawa's, both result in estimators "efficient" with respect to relative efficiency, so that his estimators and those given here agree. The method given here is more amenable to numerical calculation and is somewhat differently motivated.

Since estimates from one and two quantiles bear repeating, we begin the discussion with one quantile. The mean (but not the standard) deviation can be estimated from a single quantile. In this case, as is well known, one chooses the median of the sample, the quantile of order 0.5, for the estimator $\hat{\mu} = z_{0.5}$. The estimate is unbiased with

$$\sigma^2(\hat{\mu}) = \frac{(0.5)^2}{n f^2(0)} = \frac{1.571}{n}$$

so that

$$\text{Eff}(\hat{\mu}) = 0.637$$

To obtain an estimate of σ as well as one for μ, at least two quantiles are needed. Hence, denoting by z_1 and z_2 the sample quantiles of order p_1 and p_2 such that $p_1 < p_2 = 1 - p_1$, the following maximum likelihood estimators are obtained (assuming z_1 and z_2 jointly normal):

$$\hat{\mu} = \frac{1}{2}(z_1 + z_2)$$

$$\hat{\sigma} = \frac{z_2 - z_1}{\zeta_2 - \zeta_1} = \frac{z_2 - z_1}{2\zeta_2}$$

$$\sigma^2(\hat{\mu}) = \frac{1}{4}(\sigma_1^2 + \sigma_2^2 + 2\sigma_1\sigma_2\rho_{12}) =$$

$$\frac{2p_1(1-p_1)}{4n f_1^2} + \frac{2p_1(1-p_1)}{4n f_1^2} \cdot \frac{p_1}{1-p_1} = \frac{p_1}{2n f_1^2}$$

$$\sigma^2(\hat{\sigma}) = \frac{1}{4\zeta_1^2}(\sigma_1^2 + \sigma_2^2 - 2\,\sigma_1\,\sigma_2\,\rho_{12}) = \frac{p_1(1-2p_1)}{2n\,\zeta_1^2\,f_1^2}$$

Minimizing the two variances independently results in minimum-variance unbiased estimators:

$$\hat{\mu} = \frac{1}{2}(z_1 + z_2)$$

where z_1 and z_2 are of orders

$$p_1 = 0.2709$$

$$p_2 = 0.7291$$

$$\hat{\sigma} = 0.337\,(z_2 - z_1)$$

where z_1 and z_2 are of orders

$$p_1 = 0.0694$$

$$p_2 = 0.9306$$

The efficiencies are

$$\text{eff}\,(\hat{\mu}) = 0.810$$

$$\text{eff}\,(\hat{\sigma}) = 0.652$$

To obtain estimates from more than two quantiles, one may use the results derived from considering the following more general problem.

Suppose one has s unbiased estimates, $x_1, x_2 \cdots x_s$ of a population parameter α such that the x_i are normally distributed with variances σ_i^2 and correlations ρ_{ij}, and one wishes to obtain a new unbiased estimate $\hat{\alpha}\,(x_1, x_2 \cdots x_s)$ of α as a *linear* combination of the x_i, say

$$\hat{\alpha} = \sum_{i=1}^{s} c_i\,x_i$$

$$\sum_{i=1}^{s} c_i = 1$$

determining the c_i so that the variance of α is minimized. Employing maximum-likelihood estimation theory, one forms the likelihood function $L(x_1 \cdots x_s;\alpha)$, the joint density function of the x_i, sets $\dfrac{\partial L}{\partial \alpha} = 0$ and solves for α. This provides the estimate $\hat{\alpha}$ with the required minimum variance. Accordingly, one has

$$L(x_1 \cdots x_s;\alpha) = \frac{1}{(2\pi)^{s/2}\,\sigma_1,,,\sigma_s\,\sqrt{\Delta}}$$

$$\exp\left[-\frac{1}{2\Delta}\sum_{i=1}^{s}\sum_{j=1}^{s}\frac{b_{ij}}{\sigma_i\,\sigma_j}(x_i - \alpha)(x_j - \alpha)\right]$$

where Δ denotes the determinant of the correlation matrix (ρ_{ij}), and

$$\frac{(b_{ij})}{\Delta} = (\rho_{ij})^{-1}$$

$$b_{ij} = b_{ji}$$

$$V = \frac{\partial\, ln\, L}{\partial \alpha} = \frac{1}{\Delta}\sum_{i=1}^{s}\sum_{j=1}^{s}(x_i - \alpha)\frac{b_{ij}}{\sigma_i\,\sigma_j}$$

Solving $V = 0$ for $\hat{\alpha}$ gives for the solution $\hat{\alpha}$:

$$\hat{\alpha} = \frac{\displaystyle\sum_{i=1}^{s} x_i\left(\sum_{j=1}^{s}\frac{b_{ij}}{\sigma_i\,\sigma_j}\right)}{\displaystyle\sum_{i=1}^{s}\sum_{j=1}^{s}\frac{b_{ij}}{\sigma_i\,\sigma_j}} \tag{1}$$

The variance of $\hat{\alpha}$ can be obtained directly by observing that the Cramér-Rao inequality for an unbiased estimate,

$$\text{var}\,\hat{\alpha} \geqq \frac{1}{E(V^2)}$$

becomes an equality when the correlation between V and α is $\pm$ 1, since (Ref. 7)

$$E(V) = 0,\ \text{cov}\,(V,\alpha) = 1$$

so that when the correlation is ± 1,

$$\text{var}\,\hat{\alpha} = \frac{[\text{cov}\,(V,\alpha)]^2}{\text{var}\,V} = \frac{1}{\text{var}\,V} = \frac{1}{E(V^2)}$$

Since V is a linear function of $\hat{\alpha}$, namely

$$V = \frac{\sum_{i=1}^{s}\sum_{j=1}^{s}\frac{b_{ij}}{\sigma_i\sigma_j}}{\Delta}\hat{\alpha} - \frac{\alpha}{\Delta}\sum_{i=1}^{s}\sum_{j=1}^{s}\frac{b_{ij}}{\sigma_i\sigma_j}$$

the condition on the correlation between V and $\hat{\alpha}$ is satisfied. Hence var $\hat{\alpha}$ is given by

$$\text{var}\,\alpha = \left(E(V^2)\right)^{-1} = -\left[E\left(\frac{\partial V}{\partial \alpha}\right)\right]^{-1} = \frac{\Delta}{\sum_{i=1}^{s}\sum_{j=1}^{s}\frac{b_{ij}}{\sigma_i\sigma_j}}$$

From this it can be seen that $\hat{\alpha}$ as a linear unbiased estimator is an "efficient estimator."

An estimator for μ may now be constructed (by reduction to the case of two estimates) from *three* quantiles by letting $x_1 = z_2$ and $x_2 = \frac{1}{2}(z_1 + z_3)$ in Eq. (1), where z_2 is of order 0.5 and z_1 and z_3 are of orders p_1 and $p_3 = 1 - p_1 > p_1$. For this case, after some calculation, one has

$$\sigma^2_{x_1} = \frac{1}{4nf_2^2}, \quad \sigma^2_{x_2} = \frac{p_1}{2nf_1^2}, \quad \rho_{x_1x_2} = \sqrt{2p_1}$$

$$\hat{\mu} = \frac{4z_2(p_1f_2^2 - p_1f_2f_1) + (z_1 + z_3)(f_1^2 - 2p_1f_2f_1)}{2(f_1^2 + 2p_1f_2^2 - 4p_1f_2f_1)}$$

$$\sigma^2(\hat{\mu}) = \frac{p_1(1 - 2p_1)}{2n(f_1^2 + 2p_1f_2^2 - 4p_1f_2f_1)}$$

The unbiased estimator with minimum variances is given by

$$\hat{\mu} = 0.416\,z_2 + 0.292\,(z_1 + z_3)$$

where z_1, z_2 and z_3 are of orders $p_1 = 0.1587$, $p_2 = 0.5$, $p_3 = 0.8413$

$$\text{eff}\,(\hat{\mu}) = 0.882$$

Only $p_1 + p_3 = 1$ (symmetric quantiles) were considered, since for maximum efficiency this is the case (Ref. 2).

Estimates for both μ and σ can be obtained for even $k \geqq 4$ by the same procedure. For the mean, using four quantiles, let

$$x_1 = \frac{1}{2}(z_1 + z_4)$$

$$x_2 = \frac{1}{2}(z_2 + z_3)$$

where z_i is of order p_i; $i = 1, 2, 3, 4$, and $p_1 + p_4 = p_2 + p_3 = 1$, $0 < p_1 < p_2 < p_3 < p_4 < 1$. Then

$$\sigma^2_{x_1} = \frac{p_1}{2nf_1^2}$$

$$\sigma^2_{x_2} = \frac{p_2}{2nf_2^2}$$

$$\rho_{x_1x_2} = \sqrt{\frac{p_1}{p_2}}$$

From Eq. (1), one has

$$\hat{\mu} = \frac{(z_1 + z_4)(p_2f_1^2 - p_1f_1f_2) + (z_2 + z_3)(p_1f_2^2 - p_1f_1f_2)}{2(p_1f_2^2 + p_2f_1^2 - 2p_1f_1f_2)}$$

$$\sigma^2(\hat{\mu}) = \frac{p_1p_2 - p_1^2}{2n(p_1f_2^2 + p_2f_1^2 - 2p_1f_1f_2)}$$

To estimate σ, from four quantiles, one must use as found

$$x_1 = \frac{z_4 - z_1}{2\zeta_4}$$

$$x_2 = \frac{z_3 - z_2}{2\zeta_3}$$

$$\sigma^2_{x_1} = \frac{p_1(1 - 2p_1)}{2n\zeta_1^2 f_1^2}$$

$$\sigma_{x_2} = \frac{p_2(1 - 2p_2)}{2n\zeta_2^2 f_2^2}$$

$$\rho_{x_1x_2} = \sqrt{\frac{p_1(1 - 2p_2)}{p_2(1 - 2p_1)}}$$

Then

$$\hat{\sigma} = \frac{\dfrac{z_4 - z_1}{2\zeta_4}\left[p_2(1-2p_2)\zeta_1^2 f_1^2 - p_1(1-2p_2)\zeta_1\zeta_2 f_1 f_2\right] + \dfrac{p_1(z_3 - z_2)}{2\zeta_3}\left[(1-2p_1)\zeta_2^2 f_2^2 - (1-2p_2)\zeta_1\zeta_2 f_1 f_2\right]}{p_1(1-2p_1)\zeta_2^2 f_2^2 + p_2(1-2p_2)\zeta_1^2 f_1^2 - 2p_1(1-2p_2)\zeta_1\zeta_2 f_1 f_2}$$

and

$$\sigma^2(\hat{\sigma}) = \frac{p_1(1-2p_2)\left[p_2(1-2p_1) - p_1(1-2p_2)\right]}{2n\left[p_1(1-2p_1)\zeta_2^2 f_2^2 + p_2(1-2p_2)\zeta_1^2 f_1^2 - 2p_1(1-2p_2)\zeta_1\zeta_2 f_1 f_2\right]}$$

By varying p_1 and p_2 using a digital computer the unbiased minimum variance estimators of μ and σ were found to be:

$$\hat{\mu} = 0.198\,(z_1 + z_4) + 0.302\,(z_2 + z_3)$$

where z_1, z_2, z_3, z_4 are of orders $p_1 = 0.1068$, $p_2 = 0.3512$, $p_3 = 0.6488$, $p_4 = 0.8932$, respectively,

and

$$\hat{\sigma} = 0.116\,(z_4 - z_1) + 0.236\,(z_3 - z_2)$$

where z_1, z_2, z_3, z_4 are of orders $p_1 = 0.0230$, $p_2 = 0.1271$, $p_3 = 0.8729$, $p_4 = 0.9770$

$$\text{eff}\,(\hat{\mu}) = 0.920$$

$$\text{eff}\,(\hat{\sigma}) = 0.824$$

Tables 1 and 2 summarize the results obtained for $6 \leqq k \leqq 20$. Since it is assumed that *both* μ and σ are unknown, estimators of σ are given for an *even* number of quantiles only. Although some of the expressions given by Ogawa (Ref. 2, p. 279) as optimum estimates of μ are slightly biased, there is no significant difference between his estimators and those given in Table 1. (However, the coefficient of his estimator of σ (Ref. 2, p. 282) using two quantiles is too large by a factor of two.) We have been so far unable to prove that the optimum estimates for σ use symmetric quantiles, but only these have been considered in the tables. Symmetric quantiles have the further advantage of being less sensitive to departures from normality which result in skew distributions, skewed *away* from the location of nonsymmetric quantiles.

The minimization of the sum of variances $\sigma^2(\hat{\mu}) + \sigma^2(\hat{\sigma})$ for $k = 4$ results in the following estimators, both using the same quantiles; the details are omitted:

$$\hat{\mu} = 0.141\,(z_1 + z_4) + 0.359\,(z_2 + z_3)$$

$$\hat{\sigma} = 0.258\,(z_4 - z_1) + 0.205\,(z_3 - z_2)$$

where z_1, z_2, z_3, z_4 are of orders $p_1 = 0.0668$, $p_2 = 0.2912$, $p_3 = 0.7088$, $p_4 = 0.9332$,

$$\text{eff}\,(\hat{\mu}) = 0.908$$

$$\text{eff}\,(\hat{\sigma}) = 0.735$$

Minimizing the linear combination $\sigma^2(\hat{\mu}) + 2\sigma^2(\hat{\sigma})$ for $k = 4$ results in the following estimators, both using the same quantiles

$$\hat{\mu} = 0.093\,(z_1 + z_4) + 0.408\,(z_2 + z_3)$$

$$\hat{\sigma} = 0.171\,(z_4 - z_1) + 0.237\,(z_3 - z_2)$$

where z_1, z_2, z_3, z_4 are of orders $p_1 = 0.036$, $p_2 = 0.212$, $p_3 = 0.788$, $p_4 = 0.964$,

$$\text{eff}\,(\hat{\mu}) = 0.854$$

$$\text{eff}\,(\hat{\sigma}) = 0.796$$

Tables 3–8 give the results for $6 \leqq k \leqq 20$ and $b = 1,2,3$.

The decision to minimize the linear combination $\sigma^2(\hat{\mu}) + b\sigma^2(\hat{\sigma})$ in order to obtain suboptimum sets of quantiles to be used in estimating both μ and σ, is admittedly an arbitrary one. For $b = 1$ the efficiency of $\hat{\mu}$ is scarcely affected when compared to that obtained using the optimal set of quantiles for $\hat{\mu}$ only, while the efficiency of $\hat{\sigma}$, although considerably improved for small values of k over what can be achieved using fewer quantiles, suffers a greater loss than does the efficiency of $\hat{\mu}$

with this particular compromise. If the variation of $\hat{\sigma}$ is the important consideration, increasing the value of b gives different suboptimal sets of quantiles for which the efficiency of $\hat{\sigma}$ is improved. (Ogawa maximizes the gain in Fisher's information; it is hard to relate this to an equivalent choice of b.)

For increasing values of k, the two optimum extreme quantile values, those of orders p_1 and p_k, move farther out on the tails of the distribution. Fig. 2 exhibits the monotone decreasing property of the optimum values of p_1. Although from a theoretical viewpoint this fact is of little consequence, from practical considerations there are two major objections to this behavior of p_1 and p_k. First, since n is never infinite, the true distributions of the sample quantiles are only approximately normal and, more importantly, the deviation from normality becomes more pronounced the farther the quantiles move out on the tails of the distribution. Second, the "normal" distributions that one encounters in practical situations are

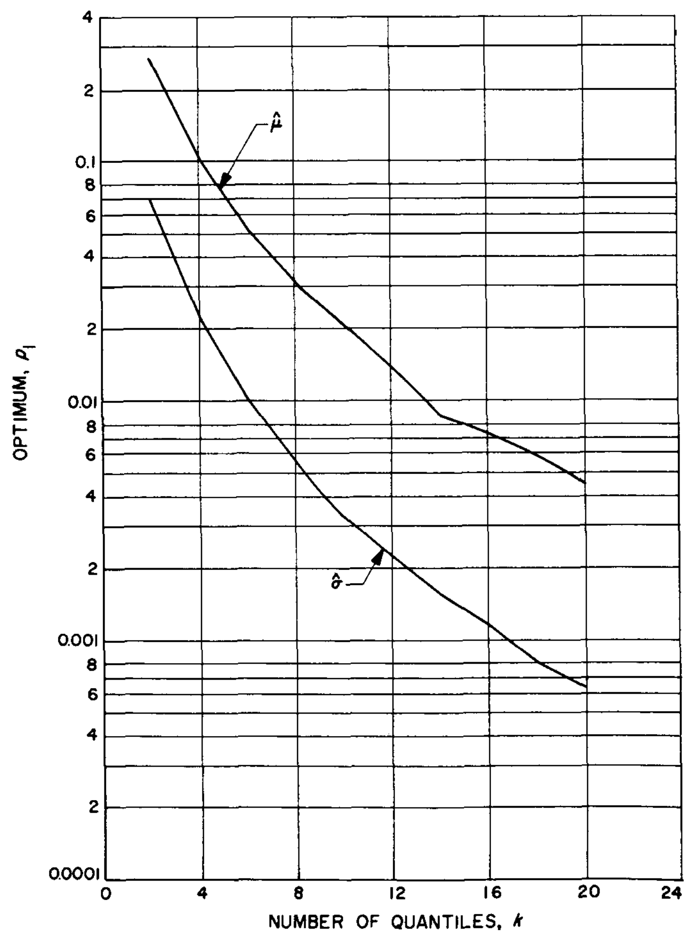

Fig. 2. Optimum values of p_1 for estimators $\hat{\mu}$ and $\hat{\sigma}$ obtained by minimizing $\sigma^2(\hat{\mu})$ and $\sigma^2(\hat{\sigma})$

very often only approximately normal with deviations from normality greater out on the tails than toward the center of the distribution. It is thus important on both counts to investigate the effect on the efficiency of the estimators when optimum and suboptimum estimates of μ and σ are obtained when p_1 is restricted to be not less than some specified value. If the loss in efficiency is not excessive it may well prove advantageous to adopt the cautious policy of restricting the value of p_1 and thus avoid or limit a bias in the estimates of μ and σ due either to a sufficiently large deviation from the assumed normality of the distribution of the extreme quantiles or to the erratic behavior out on the tail of an approximately normal parent distribution. Tables 9–16 give optimum and suboptimum estimators of μ and σ when p_1 is restricted to be not less than 0.01. Tables 17–24 give similar results for p_1 not less than 0.025. For comparison purposes, Tables 25–27 list the efficiencies of the estimators for $6 \leqq k \leqq 20$, k even, under the various conditions and restrictions. It is readily seen from these tables that although the restrictions on p_1 affect the efficiencies of $\hat{\sigma}$ to a greater extent than those of $\hat{\mu}$, nevertheless efficiencies greater than 0.90 can be achieved for suboptimum estimators of σ for $k \geqq 10$ when p_1 is restricted to be not less than 0.025. In fact, for this case very little is gained by using more than 10 quantiles!

If the population is distributed with mean μ and variance σ^2 (instead of being the *unit* normal), all the optimal quantiles and estimators given above for μ and σ are the same, as is necessary if the system is to be *used* for unknown μ, σ. This is obvious in the case of $\hat{\mu}$; it follows in the case of $\hat{\sigma}$ from the following observation. In general, using two quantiles,

$$\hat{\sigma} = \frac{z_2 - z_1}{2\zeta_2^*}$$

where z_1 and z_2 are of orders p_1 and $p_2 = 1 - p_1 > p_1$; the corresponding population quantiles are ζ_1 and ζ_2, while ζ_2^* is the corresponding quantile of the unit normal distribution. One then sees that

$$E(\hat{\sigma}) = \frac{\zeta_2 - \zeta_1}{2\zeta_2^*} = \frac{\sigma\zeta_2^* + \mu - \sigma\zeta_1^* - \mu}{2\zeta_2^*} = \sigma$$

since $\zeta_1^* = -\zeta_2^*$.

This proves the assertion, and shows that the system can actually be used with the calculated quantiles and efficiencies as above.

III. A GOODNESS-OF-FIT TEST

A test which decides on the basis of criteria fixed beforehand whether or not it is reasonable to regard a set of n observed values of a random variable as coming from a population with a specified probability distribution is said to be a goodness-of-fit test. A number of tests of this nature exist at present when all of the sample values are available. But when data must be transmitted from a space probe to Earth and a high data-compression ratio is desired, a goodness-of-fit test which can be applied to a *small* amount of received data is highly advantageous. Such a test, based on only four sample quantiles (and a large value of n), is presented below. Moreover, subject to the condition that the density function of the hypothesized distribution possesses a continuous derivative in some neighborhood of each quantile value (Ref. 5), the test is (asymptotically) *distribution* free.

Let H_0 denote the null hypothesis that the parent population has a probability distribution with density function $f(x)$ (which meets the above condition on its derivative), and let H_1 denote the (composite) alternative hypothesis that the density function is not $f(x)$. Let ζ_1, ζ_2, ζ_3, ζ_4 be four quantiles of orders p_1, p_2, p_3, p_4, and let z_1, z_2, z_3, z_4 denote the corresponding sample quantiles. A criterion on the basis of which one may accept or reject H_0 will be established as a function of the sample quantiles z_i, so that the probability of an error of the first kind will be equal to a given ε, the significance level of the test. (Then the p_i could be chosen to minimize the probability of an error of the second kind under various simple alternatives.) The analysis will be based as in Section II, on the limiting distributions and moments of the z_i.

Since the random variables z_1, z_2, z_3, z_4 are not independent, it is convenient to transform them into four new random variables, x_1, x_2, x_3, x_4, under the following triangular transformation, by the Gram-Schmidt orthogonalization procedure familiar from the theory of orthogonal spaces.

So let

$$x_1 = \frac{1}{\sigma_1}(z_1 - \zeta_1)$$

$$x_2 = (1 - \rho_{12}^2)^{-1/2}\left[\frac{1}{\sigma_2}(z_2 - \zeta_2) - \rho_{12}x_1\right]$$

$$x_3 = \beta_1\left[\frac{(1-\rho_{12}^2)^{1/2}}{\sigma_3}(z_3 - \zeta_3) - (\rho_{23} - \rho_{12}\rho_{13})x_2 - \rho_{13}(1-\rho_{12}^2)^{1/2}x_1\right]$$

$$x_4 = \alpha_1\left[\frac{1}{\sigma_4}(z_4 - \zeta_4) - \beta_3 x_3 - \beta_2 x_2 - \rho_{14}x_1\right]$$

where σ_i^2 is the variance of z_i and ρ_{ij} the correlation between z_i and z_j under H_0.

Here

$$\beta_1 = (1 - \rho_{12}^2 - \rho_{13}^2 - \rho_{23}^2 + 2\rho_{12}\rho_{13}\rho_{23})^{-1/2}$$

$$\beta_2 = (\rho_{24} - \rho_{12}\rho_{14})(1 - \rho_{12}^2)^{-1/2}$$

$$\beta_3 = \beta_1\left[(\rho_{34} - \rho_{13}\rho_{14})(1-\rho_{12}^2)^{1/2} - (1-\rho_{12}^2)^{-1/2}(\rho_{23} - \rho_{12}\rho_{13})(\rho_{24} - \rho_{12}\rho_{14})\right]$$

$$\alpha_1 = (1 - \beta_2^2 - \beta_3^2 - \rho_{14}^2)^{-1/2}$$

It is not difficult to verify that, *under the null hypothesis*, the x_i, $i = 1, 2, 3, 4$ are normally distributed random variables, each with zero mean and unit variance, and *independent*. Thus the likelihood function is the density function of the joint distribution of the x_i, given by

$$L(x_1, x_2, x_3, x_4) = \frac{1}{(2\pi)^2}\exp\left(-\frac{1}{2}\sum_{i=1}^{4} x_i^2\right)$$

which has a maximum value of $(2\pi)^{-2}$. A critical or rejection region can now be designated as the interval $0 < L < A$, $A < (2\pi)^{-2}$, where A is determined such that, given H_0, the probability of $L(x_1, x_2, x_3, x_4)$ lying in this interval is equal to ε. However, since $L(x_1, x_2, x_3, x_4)$ is a monotone function of $y = \sum_{i=1}^{4} x_i^2$, and $y = 0$ when $L = (2\pi)^{-2}$, $y \to \infty$ when $L \to 0$, a critical region of the form $0 < L < A$ is equivalent to a critical region $y > B$, where B is determined so that the probability of $y > B$ equals ε. Now, y has the chi-square distribution with four degrees of freedom, so that its density function $K(y)$ is given by

$$K(y) = \frac{1}{4}\, y\, e^{-y/2}$$

$$y \geqq 0$$

Hence, B is the solution to the equation

$$\int_0^B K(y)\, dy = 1 - \varepsilon$$

For $\varepsilon = 0.05$, B = 9.5; and for $\varepsilon = 0.01$, B = 13.3.

To test H_0, therefore, transform the observed values of z_i (of order p_i) $i = 1, 2, 3, 4$, to the x_i by means of the above transformation; if $\sum_{i=1}^{4} x_i^2 < B$, accept H_0. If $\sum_{i=1}^{4} x_i^2 > B$, reject H_0. The decision will be made on a significance level of ε.

In practical situations, the null hypothesis is often made on the form of the density function which may be a function of several *unknown* parameters. For example, one may wish to test the hypothesis that the parent population is normally distributed with unknown mean and variance. In order to carry out this non-studentized test, estimates of the mean μ and variance σ^2 of the population are required. For this purpose the estimators obtained previously by minimizing $\sigma^2(\hat{\mu}) + \sigma^2(\hat{\sigma})$ may be used:

$$\hat{\mu} = 0.141\,(z_1 + z_4) + 0.359\,(z_2 + z_3)$$

$$\hat{\sigma} = 0.258\,(z_4 - z_1) + 0.205\,(z_3 - z_2)$$

where

$$p_1 = 0.0668,\ p_2 = 0.2912,\ p_3 = 0.7088,\ p_4 = 0.9332$$

Although the estimators are derived on the basis of a normally distributed parent population, they are relatively insensitive to deviations from normality. Replacing μ and σ by $\hat{\mu}$ and $\hat{\sigma}$, ζ_i and $f(\zeta_i)$ can be determined, where each ζ_i is of the same order as the corresponding z_i used in the estimators of μ and σ. Thus to test the hypothesis that a set of n sample values has been taken from a normally distributed population with unknown mean and variance, using four quantiles, the above estimators can be used. The transformation to independent variables for this special case then becomes

$$x_1 = \frac{0.519\sqrt{n}}{\hat{\sigma}}\,(z_1 - \mu_1)$$

$$x_2 = 0.830\frac{\sqrt{n}}{\hat{\sigma}}\,(z_1 - \mu_2) - 0.459\,x_1$$

$$x_3 = 0.827\frac{\sqrt{n}}{\hat{\sigma}}\,(z_3 - \mu_3) - 0.410\,x_2 - 0.187\,x_1$$

$$x_4 = 0.571\frac{\sqrt{n}}{\hat{\sigma}}\,(z_4 - \mu_4) - 0.418\,x_3 - 0.170\,x_2 - 0.079\,x_1$$

where

$$\mu_1 = \hat{\mu} - 1.50\hat{\sigma}$$

$$\mu_2 = \hat{\mu} - 0.55\hat{\sigma}$$

$$\mu_3 = \hat{\mu} + 0.55\hat{\sigma}$$

$$\mu_4 = \hat{\mu} + 1.50\hat{\sigma}$$

The above test was applied to two sets of samples, each taken from non-normally distributed populations in order to test for normality. In case A, the parent population was bimodal with a density function given by

$$f_A(x) = \frac{0.2}{\sqrt{2\pi}}\, e^{\frac{-x^2}{2}} + \frac{0.8}{3\sqrt{2\pi}}\, e^{-1/2\left(\frac{x-9}{3}\right)^2}$$

500 sample values were generated from entries taken from a table of random numbers. The optimal set of four sample quantiles for the joint estimation of μ and σ was determined and used to obtain the results $\hat{\mu} = 7.19$, $\hat{\sigma} = 4.35$, as compared to $\mu = 7.2$, $\sigma = 4.51$. After the transformation to independent variables *under the null hypothesis that the population was normally distributed,* the result $\sum_{i=1}^{4} x_i^2 = 39.0$ was obtained! Since this means that it is highly improbable that H_0 is true, the result is quite satisfactory.

With respect to case B, the parent population was also bimodal with a density function given by

$$f_B(x) = \frac{0.3}{\sqrt{2\pi}}\, e^{-x^2/2} + \frac{0.7}{3\sqrt{2\pi}}\, e^{-1/2\left(\frac{x-4}{3}\right)^2}$$

The results in this case with $n = 250$ were: $\hat{\mu} = 2.69$, $\hat{\sigma} = 3.09$ as compared to $\mu = 2.8$, $\sigma = 3.15$ and

$\sum_{i=1}^{4} x_i^2 = 22.5$. Thus, for this case H_0 would also be rejected at significance levels of 0.05 and 0.01. The null hypothesis that the second set of samples came from a population with the true density function $f_B(x)$ was then tested. For this case, $\sum_{i=1}^{4} x_i^2 = 1.23$, so that the hypothesis would be *accepted.*

The power of the test, that is, the probability of rejecting H_0 when it is false, obviously depends on the nature of the true distribution of the population (as well as on the significance level). In order to determine the power of the test relative to a specific alternate density function, one must find the probability that $\sum_{i=1} x_i^2 > B$ *under the simple alternative hypothesis* H_1, when the x_i are random variables resulting from transforming the z_i *under* H_0. Consequently, although the x_i are still normally distributed under H_1, they are in general correlated and have means and variances other than 0 and 1, respectively. Under these conditions, the problem of determining the distribution of $y = \sum_{i-1} x_i^2$ under H_1 is quite difficult. Accordingly, in order to estimate the power of the test by "Monte Carlo" methods when H_0 is the hypothesis that the population in case B is distributed with the given normal distribution and H_1 the hypothesis that the population has a distribution with the true density function $f_B(x)$ given above, the z_i were transformed to the x_i *under* H_0, and then the first two moments and correlation of the x_i were computed under H_1. Then 152 sets of the four x_i were generated from a table of random numbers, and for each set $\sum_{i=1} x_i^2$ was computed. In 99.3% of the cases, $\sum_{i=1} x_i^2 > 9.5$, and in 95.4% of the cases $\sum_{i=1} x_i^2 > 13.3$. Thus the power is quite high.

The power of the test may be increased by using more than four quantiles. For example, the median may also be used. If five quantiles are used, the transformations for the first four remain the same (a useful property of the Gram-Schmidt procedure) and the fifth transformation is given by

$$x_5 = \alpha_2 \left(\frac{1}{\sigma_5} (z_5 - \zeta_5) - \gamma_4 x_4 - \gamma_3 x_3 - \gamma_2 x_2 - \rho_{15} x_1 \right)$$

where

$$\gamma_2 = (1 - \rho_{12}^2)^{-1/2} (\rho_{25} - \rho_{12}\rho_{15})$$

$$\gamma_3 = \beta_1 \left[(\rho_{35} - \rho_{13}\rho_{15})(1 - \rho_{12})^{1/2} - (\rho_{23} - \rho_{12}\rho_{13}) (\rho_{25} - \rho_{12}\rho_{15})(1 - \rho_{12}^2)^{-1/2} \right]$$

$$\gamma_4 = \alpha_1 \{ \rho_{45} - \beta_3 \beta_1 \left[\rho_{35} (1 - \rho_{12})^{1/2} - (\rho_{23} - \rho_{12}\rho_{13}) (\rho_{25} - \rho_{12}\rho_{15})(1 - \rho_{12}^2)^{-1/2} - \rho_{13}\rho_{15}(1 - \rho_{12}^2)^{1/2} \right] - \beta_2 (1 - \rho_{12}^2)^{-1/2} (\rho_{25} - \rho_{12}\rho_{15}) - \rho_{14}\rho_{15} \}$$

$$\alpha_2 = (1 - \gamma_4^2 - \gamma_3^2 - \gamma_2^2 - \rho_{15}^2)^{-1/2}$$

A generalization of this procedure to m quantiles will now be given. Let $z_1', z_2' \cdots z_m'$ denote m sample quantiles of orders $p_1, p_2 \cdots p_m$ and let $\zeta_1', \zeta_2' \cdots \zeta_m'$ denote the corresponding population quantiles. First, let

$$z_i = \frac{z_i' - \zeta_i'}{\sigma_i}, i = 1, 2, \cdots m$$

Then let

$$x_1 = z_1$$

$$x_2 = a_{21} x_1 + a_{22} z_2$$

$$x_3 = a_{31} x_1 + a_{32} x_2 + a_{33} z_3$$

- -

$$x_k = a_{k1} x_1 + a_{k2} x_2 + \cdots + a_{k,k-1} x_{k-1} + a_{kk} z_k$$

- -

$$x_m = a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{m,m-1} x_{m-1} + a_{mm} z_m$$

The set of coefficients $\{a_{ij}\}$ will be determined recursively so as to satisfy the following conditions:

$$\left.\begin{array}{l} (1)\ E(x_i) = 0 \\ (2)\ E(x_i^2) = 1 \\ (3)\ E(x_i x_j) = 0, i \neq j \end{array}\right\} \quad i,j = 1, 2, \cdots m$$

The first condition, $E(x_i) = 0$, was satisfied by normalizing the z_i'. Applying conditions (2) and (3) one has

$$E(x_2^2) = a_{21}^2 + a_{22}^2 + 2a_{21} a_{22} \rho_{12} = 1$$

$$E(x_1 x_2) = a_{21} + a_{22}\rho_{12} = 0$$

Solving for a_{21} and a_{22}, one obtains:

$$a_{22} = (1 - \rho_{12}^2)^{-1/2}$$

$$a_{21} = -\rho_{12}\, a_{22}$$

To determine the coefficients of x_3, one has:

$$E(x_3^2) = a_{31}^2 + a_{32}^2 + a_{33}^2 + 2a_{33}\left[a_{31}\,\rho_{13} + a_{32}\,(a_{21}\,\rho_{13} + a_{22}\,\rho_{23})\right] = 1$$

$$E(x_1 x_2) = a_{31} + a_{33}\,\rho_{13} = 0$$

$$E(x_2 x_3) = a_{32} + a_{33}\,(a_{21}\,\rho_{13} + a_{22}\,\rho_{23}) = 0$$

Solving for the a_{3i}, $i = 1, 2, 3$, one obtains:

$$a_{33} = \left[1 - \rho_{13}^2 - (a_{21}\,\rho_{13} + a_{22}\rho_{23})^2\right]^{-1/2}$$

$$a_{31} = -\rho_{13}\, a_{33}$$

$$a_{32} = -(a_{21}\,\rho_{13} + a_{22}\,\rho_{23})\, a_{33}$$

Continuing in this manner, one obtains, in general,

$$a_{kk} = \left[1 - \rho_{1k}^2 - (a_{21}\,\rho_{1k} + a_{22}\,\rho_{2k})^2 - (a_{31}\,\rho_{1k} + a_{32}\,\rho_{2k} + a_{33}\,\rho_{3k})^2 - \cdots - (a_{k-1,1}\,\rho_{1k} + a_{k-1,2}\,\rho_{2k} + \cdots + a_{k-1,k-1}\,\rho_{k-1,k})^2\right]^{-1/2}$$

$$a_{k1} = -\rho_{1k}\, a_{kk}$$

$$a_{k2} = -(a_{21}\,\rho_{1k} + a_{22}\,\rho_{2k})\, a_{kk}$$

$$a_{k3} = -(a_{31}\,\rho_{1k} + a_{32}\,\rho_{2k} + a_{33}\,\rho_{3k})\, a_{kk}$$

$$a_{k,k-1} = -(a_{k-1,1}\,\rho_{1k} + a_{k-1,2}\,\rho_{2k} + \cdots + a_{k-1,k-1}\,\rho_{k-1,k})\, a_{kk}$$

Expressions for the transformation of additional quantiles can now be obtained with a corresponding increase in the complexity of the expressions.

IV. A GOODNESS-OF-FIT TEST DESIGNED FOR HIGH POWER AGAINST BIMODAL DISTRIBUTION

A set of n observations of a random variable obtained as a result of a space experiment may contain sample values taken from *two* distinct populations rather than from a single population as one might ordinarily expect. For example, one may be interested in the "energy spectrum" of incoming particles. If the samples are all taken from a single source it is reasonable to assume that the parent population is unimodal, i.e., that the population density has a unique local maximum. But if the samples come from either of two possible sources, with probability p that a sample is taken from one source and probability $1 - p$ that a sample is taken from the other then the observations can be regarded as a set of sample values of a random quantity with a probability density function given by $f(x) = pf_1(x) + (1 - p) f_2(x)$, where p is the probability of the first source, $1 - p$ the probability of the second source, f_1 the density function of the first source, and f_2 the density function of the second source. This case could occur if there are two sources of radiation, say solar electrons and cosmic ray electrons. Then f may not be unimodal even if f_1 and f_2 were.

If doubt exists as to which is the true situation, one would like to apply a goodness-of-fit test on the basis of which one may decide whether the parent population is actually unimodal or whether its distribution is described by the above density function $f(x)$, which is very likely to be bimodal. We shall consider only the cases where f_1, f_2 are density functions of normal populations. Although the test previously presented is applicable to a general class of probability distributions including those under consideration here, it is in general desirable to apply additional tests when they are available. Hence another goodness-of-fit test, also based on four sample quantiles and a large value of n, will be presented.

Let H_0 denote the null hypothesis that a set of n observations came from a normally distributed population, with (unknown) mean μ and (unknown) variance σ^2. Let z_1', z_2', z_3', z_4' be four sample quantiles of orders p_1, p_2, p_3, p_4, where $p_1 + p_4 = p_2 + p_3 = 1$, $0 < p_1 < p_2 < p_3 < p_4 < 1$, and let $\zeta_1', \zeta_2', \zeta_3', \zeta_4'$ denote the corresponding quantiles of the distribution. Since μ and σ^2 are unknown, the optimal estimates obtained by minimizing, say, the sum of the variances of $\hat{\mu}$ and $\hat{\sigma}$ will be used in the test to replace μ and σ, namely

$$\hat{\mu} = 0.141\,(z_1' + z_4') + 0.359\,(z_2' + z_3')$$

$$\hat{\sigma} = 0.258\,(z_4' - z_1') + 0.205\,(z_3' - z_2')$$

where $p_1 = 0.0668, p_2 = 0.2912, p_3 = 0.7088, p_4 = 0.9332$

It is convenient to first normalize the sample quantiles by means of the transformations

$$z_i = \frac{z_i' - \mu}{\sigma} \cong \frac{z_i' - \hat{\mu}}{\hat{\sigma}}, \; i = 1, 2, 3, 4$$

so that the z_i are the corresponding sample quantiles of a normal population with zero mean and unit variance. (As has been seen, $\hat{\mu}$ and $\hat{\sigma}$ are arbitrarily good estimates for μ and σ when n is large, as is the case here.)

Now let

$$y_1 = z_4 - z_3 - (z_2 - z_1) = z_4 - z_3 - z_2 + z_1$$

$$\begin{aligned} y_2 &= z_3 - z_2 - (z_4 - z_3 + z_2 - z_1) \\ &= 2z_3 - 2z_2 - z_4 + z_1 \end{aligned}$$

The following goodness-of-fit test is now applied, based on the values of y_1 and y_2 as follows: If *either* $|y_1| > k_1$ or $y_2 > k_2$, reject H_0. If $|y_1| \leqq k_1$ and $y_2 \leqq k_2$, accept H_0. If ε denotes the significance level of the test, k_1 and k_2 are determined by the relation prob $(|y_1| > k_1)$ + prob $(y_2 > k_2)$ − prob $(|y_1| > k_1, y_2 > k_2) = \varepsilon$. Table 28 gives values of k_1 and k_2 for $\varepsilon = 0.05$ and $\varepsilon = 0.01$ for various values of n, where prob $(|y_1| > k_1)$ is taken to be equal to prob $(y_2 > k_2)$.

The choice of y_1 and y_2 for applying rejection criteria is motivated by a need for a goodness-of-fit test which, when applied to a null hypothesis of normality, would be likely to have considerable power when the composite alternative hypotheses, H_1, is that the parent density is bimodal. Since $p_2 - p_1 = p_4 - p_3$, the criterion based on the value of y_1 essentially tests for *symmetry*. However, since a bimodal distribution can possess sufficient symmetry with respect to the intervals (ζ_1,ζ_2) and (ζ_3,ζ_4), so that H_0 would not be rejected solely on the basis of the value of y_1, the statistic y_2 is designed to detect the fact that a continuous bimodal density function has a local minimum between the maxima (a consequence of the theorem below); thus $\zeta_3 - \zeta_2$ is *large* in comparison with $\zeta_4 - \zeta_3$ and $\zeta_2 - \zeta_1$, if the dip is between ζ_2 and ζ_3. But if the dip occurs outside the interval (ζ_2,ζ_3) the statistic y_1 will pick out this asymmetry; hence the high power of the test.

Theorem: Let $g(x)$ be a continuous probability density function on $-\infty < x < \infty$. If there exist four values of x, say $x_1 < x_2 < x_3 < x_4$, such that

$$(1)\quad \int_{x_1}^{x_2} g(x)\,dx = \int_{x_3}^{x_4} g(x)\,dx$$

$$(2)\quad x_2 - x_1 = x_4 - x_3$$

$$(3)\quad \int_{x_2}^{x_3} g(x)\,dx < \frac{x_3 - x_2}{x_2 - x_1}\int_{x_1}^{x_2} g(x)\,dx$$

then $g(x)$ has a local minimum on the interval (x_1,x_4) (and hence is *not unimodal*).

Proof: Let $\delta = x_2 - x_1$ and let $x_3 - x_2 = k(x_2 - x_1) = k\delta$. If $g(x)$ has no local minimum on (x_1,x_4), then $g(x)$ is either constant on (x_1,x_4), or else has *exactly* one local maximum on (x_1,x_4), which maximum must occur on the interval (x_2,x_3), since $g(x)$ cannot be strictly monotone increasing or decreasing on (x_1,x_4) because of conditions (1) and (2). So let $m = \min\,[g(x_2), g(x_3)]$, say $m = g(x_2)$.

Then one has

$$\int_{x_2}^{x_3} g(x)\,dx \geqq m(x_3 - x_2) = mk\,\delta \geqq k\int_{x_1}^{x_2} g(x)\,dx$$

$$= \frac{x_3 - x_2}{x_2 - x_1}\int_{x_1}^{x_2} g(x)\,dx$$

which contradicts condition (3). This proves the theorem.

Although the converse of the theorem is of course not true, the theorem suggests that a dip in the actual density function may be detected by comparing the interval length $z_3 - z_2$ with the interval lengths $z_2 - z_1$ and $z_4 - z_3$, when H_0 cannot be rejected on the basis of the value of y_1 alone. The method of determining k_1 and k_2 will now be presented.

Under the null hypothesis of normality (and also assuming the limiting normal distribution and moments of the z_i), one has

$$E(y_1) = \zeta_4 - \zeta_3 - (\zeta_2 - \zeta_1) = 0$$

$$\sigma^2(y_1) = \sigma_4^2 + \sigma_3^2 + \sigma_2^2 + \sigma_1^2 - 2\rho_{34}\,\sigma_3\,\sigma_4 - 2\rho_{24}\,\sigma_2\,\sigma_4$$

$$+ 2\rho_{14}\,\sigma_1\,\sigma_4 + 2\rho_{23}\,\sigma_2\,\sigma_3 - 2\rho_{13}\,\sigma_1\,\sigma_3 - 2\rho_{12}\,\sigma_1\,\sigma_2 = 2\sigma_1^2$$

$$+ 2\sigma_2^2 - 4(\rho_{12} + \rho_{13})\,\sigma_1\,\sigma_2 + 2\rho_{14}\,\sigma_1^2 + 2\rho_{23}\,\sigma_2^2 = \frac{6.9155}{n}$$

$$E(y_2) = 2\zeta_3 - 2\zeta_2 - \zeta_4 + \zeta_1 = 2(\zeta_1 - 2\zeta_2) = -0.80$$

$$\sigma^2(y_2) = 4\sigma_3^2 + 4\sigma_2^2 + \sigma_4^2 + \sigma_1^2 - 8\rho_{23}\,\sigma_2\,\sigma_3 - 4\rho_{34}\,\sigma_3\,\sigma_4$$

$$+4\rho_{13}\,\sigma_1\,\sigma_3 + 4\rho_{24}\,\sigma_2\,\sigma_4 - 4\rho_{12}\,\sigma_1\,\sigma_2 - 2\rho_{14}\,\sigma_1\,\sigma_4$$

$$= 2\sigma_1^2 + 8\sigma_2^2 - 8\rho_{23}\,\sigma_2^2 - 2\rho_{14}\,\sigma_1^2$$

$$+ 8(\rho_{13} - \rho_{12})\,\sigma_1\,\sigma_2 = \frac{10.1436}{n}$$

Now the next calculation is more important:

$$E(y_1y_2) = \sigma_1^2 - \sigma_4^2 + 2\sigma_2^2 - 2\sigma_2^3 + \rho_{13}\,\sigma_1\,\sigma_3 - \rho_{24}\,\sigma_2\,\sigma_4$$

$$+ 3\rho_{34}\,\sigma_3\,\sigma_4 - 3\rho_{12}\,\sigma_1\,\sigma_2 = 0!$$

Thus, under the null hypothesis that the parent population is normal, y_1 and y_2 are, fortunately, *independent normally distributed* random variables with variances inversely proportional to the sample size n. It is an easy matter, therefore, to determine k_1 and k_2 for given values of n and ε such that prob $(|y_1| > k_1)$ = prob $(y_2 > k_2)$ $= \varepsilon_1$ (where $2\varepsilon_1 - \varepsilon_1^2 = \varepsilon$).

The above goodness-of-fit test was applied to the same two cases, f_A and f_B, considered in Section 3. In case A with $n = 500$, the results were $|y_1| = 0.736$, $y_2 = -0.864$, so that H_0 is rejected at both $\varepsilon = 0.05$ and $\varepsilon = 0.01$. In case B with $n = 250$, $|y_1| = 0.602$, $y_2 = -0.198$, so that H_0 is also rejected for both $\varepsilon = 0.05$ and $\varepsilon = 0.01$.

A set of 250 values was then taken from a table of random (unit) normal deviates, and the test applied to this set. The results were $\mu = 0.005$, $\sigma = 1.000$, $|y_1|$ $= 0.057$, $y_2 = -0.953$ so that H_0 would be accepted for both $\varepsilon = 0.05$ and $\varepsilon = 0.01$. (The excellent agreement between the sample and population moments is fortuitous and should not be construed as being typical. The estimators are not *that* good.)

To determine the power of the test for a given H_1, it is necessary to compute the first two moments of the sample quantiles *under* H_1, and then the moments of y_1 and y_2 which are still (asymptotically) normal. The power is given by $P = \text{prob}\,(|y_1| > k_1) + \text{prob}\,(y_2 > k_2) - \text{prob}\,(|y_1| > k_1, y_2 > k_2)$, where k_1 and k_2 are determined *under* H_0. If the density function under H_1 is not symmetric about its mean, y_1 and y_2 in general will not be independent so that $\text{prob}\,(|y_1| > k_1, y_2 > k_2) \neq \text{prob}\,(|y_1| > k_1)\,\text{prob}\,(y_2 > k_2)$. In this event one may use the method and tables given in Ref. 8 to compute the joint probability.

Since the deviation from normality in case B is considerably less than that for case A, the power of the test was determined for the alternative hypothesis that the population has a distribution with density function $f_B(x)$. It was found that for $\varepsilon = 0.05$, $P = 0.997$, and for $\varepsilon = 0.01$, $P = 0.938$. Thus the power is quite high in this case.

The fact that the test was devised with a specific H_0 and H_1 in mind does not preclude its use in testing other null hypotheses. In the general case the sample quantiles are still asymptotically normal so that k_1 and k_2 can be readily determined, although y_1 and y_2 will in general be correlated. Although the p_i could be chosen to maximize the power of the test under various alternatives, in practical situations involving a space probe experiment, they must be chosen in advance; since the sample quantiles are also used to estimate μ and σ, the same optimal set has been given and used in the numerical calculations.

Since the validity of the goodness-of-fit tests presented here depends strongly upon the assumed normality of the distribution of the sample quantiles used, a comparison was made between the limiting normal density function of the sample quantile z_1 of order $p_1 = 0.0668$ and the rigorous density function $g(x)$ for the case where $n = 250$ and the parent population is distributed according to the unit normal density. The quantile of order p_1 was chosen for two reasons. First, as mentioned before, the deviation from normality is more pronounced for quantiles at the tails of the parent density function so that the results obtained would be conservative. Second, the calculations would be simplified, as can be seen from the rigorous expression for the density function of z_i given by Ref. 5:

$$g(x) = \binom{n}{\mu}(n-\mu)\,[F(x)]^{\mu}[1 - F(x)]^{n-\mu-1} f(x)$$

where μ = the greatest integer equal to or less than np_1. In this case $\mu = 16$ and

$$F(x) = \frac{1}{2\pi}\int_{-\infty}^{x} e^{-t^2/2}\,dt$$

$$f(x) = F'(x)$$

The density function of the limiting normal distribution of z_1 is given by

$$h(x) = \frac{1}{0.122\sqrt{2\pi}}\, e^{-1/2\left(\frac{x+1.5}{0.122}\right)^2}$$

In Fig. 3, plots of $g(x)$ and $h(x)$ are exhibited. It is evident that $h(x)$ is an extremely close approximation to $g(x)$.

Work is continuing at the Jet Propulsion Laboratory with a view to using these data compression systems using quantiles on further planetary and deep space probes.

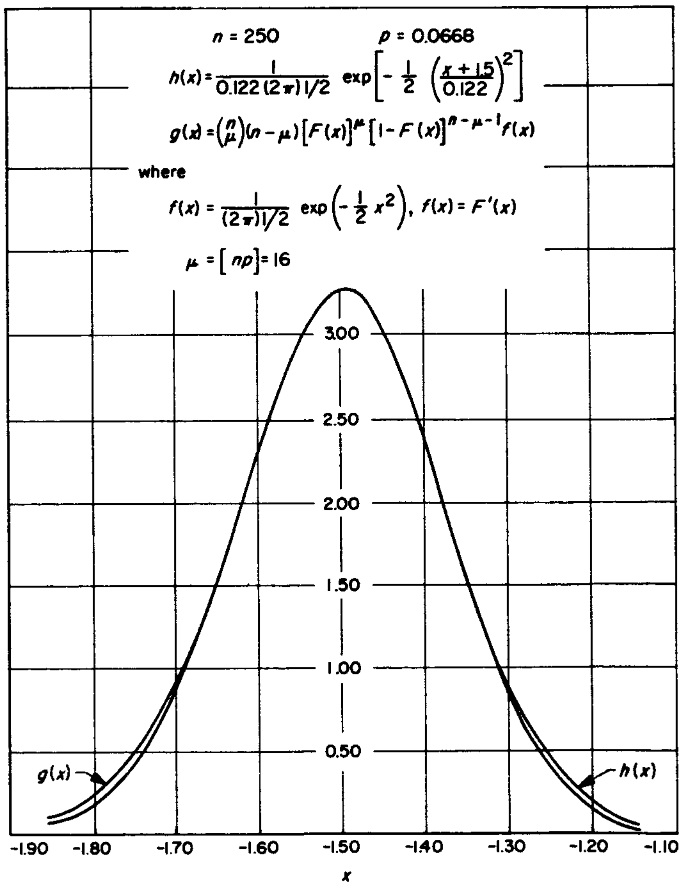

Fig. 3. A comparison between an approximate normal density function $g(x)$ and rigorous normal density function $h(x)$

Table 1. Estimators of the mean of a normal distribution with minimum variance

K	Estimators ($\hat{\mu}$)	Efficiency ($\hat{\mu}$)
6	0.0968 [z(0.0540) + z(0.9460)] + 0.1787 [z(0.1915) + z(0.8085)] + 0.2245 [z(0.3898) + z(0.6102)]	0.9560
8	0.0559 [z(0.0310) + z(0.9690)] + 0.1119 [z(0.1154) + z(0.8846)] + 0.1550 [z(0.2481) + z(0.7519)] + 0.1772 [z(0.4126) + z(0.5874)]	0.9722
10	0.0366 [z(0.0203) + z(0.9797)] + 0.0751 [z(0.0768) + z(0.9232)] + 0.1086 [z(0.1684) + z(0.8316)] + 0.1334 [z(0.2887) + z(0.7113)] + 0.1463 [z(0.4274) + z(0.5726)]	0.9808
12	0.0246 [z(0.0135) + z(0.9865)] + 0.0522 [z(0.0525) + z(0.9475)] + 0.0786 [z(0.1178) + z(0.8822)] + 0.1012 [z(0.2075) + z(0.7925)] + 0.1174 [z(0.3163) + z(0.6837)] + 0.1260 [z(0.4373) + z(0.5627)]	0.9859
14	0.0160 [z(0.00868) + z(0.99132)] + 0.0360 [z(0.0351) + z(0.9649)] + 0.0568 [z(0.0814) + z(0.9186)] + 0.0773 [z(0.1484) + z(0.8516)] + 0.0941 [z(0.2342) + z(0.7658)] + 0.1061 [z(0.3336) + z(0.6664)] + 0.1137 [z(0.4430) + z(0.5570)]	0.9892
16	0.0130 [z(0.00730) + z(0.99270)] + 0.0272 [z(0.0277) + z(0.9723)] + 0.0424 [z(0.0619) + z(0.9381)] + 0.0594 [z(0.1129) + z(0.8871)] + 0.0745 [z(0.1802) + z(0.8198)] + 0.0866 [z(0.2602) + z(0.7398)] + 0.0962 [z(0.3512) + z(0.6488)] + 0.1007 [z(0.4501) + z(0.5499)]	0.9915
18	0.0106 [z(0.00587) + z(0.99412)] + 0.0232 [z(0.0229) + z(0.9771)] + 0.0371 [z(0.0532) + z(0.9468)] + 0.0507 [z(0.0972) + z(0.9028)] + 0.0626 [z(0.1540) + z(0.8460)] + 0.0709 [z(0.2211) + z(0.7789)] + 0.0775 [z(0.2942) + z(0.7058)] + 0.0827 [z(0.3745) + z(0.6255)] + 0.0847 [z(0.4579) + z(0.5421)]	0.9931
20	0.0081 [z(0.00448) + z(0.99552)] + 0.0184 [z(0.0175) + z(0.9825)] + 0.0322 [z(0.0429) + z(0.9571)] + 0.0465 [z(0.0828) + z(0.9172)] + 0.0568 [z(0.1355) + z(0.8645)] + 0.0613 [z(0.1950) + z(0.8050)] + 0.0657 [z(0.2564) + z(0.7436)] + 0.0695 [z(0.3254) + z(0.6746)] + 0.0698 [z(0.3939) + z(0.6061)] + 0.0717 [z(0.4639) + z(0.5361)]	0.9943

Table 2. Estimators of the standard deviation of a normal distribution with minimum variance

K	Estimators ($\hat{\sigma}$)	Efficiency ($\hat{\sigma}$)
6	$0.0549\,[z(0.9896) - z(0.0104)] + 0.1244\,[z(0.9452) - z(0.0548)]$ $+ 0.1825\,[z(0.8304) - z(0.1696)]$	0.8943
8	$0.0307\,[z(0.99451) - z(0.00549)] + 0.0730\,[z(0.9714) - z(0.0286)]$ $+ 0.1168\,[z(0.9149) - z(0.0851)] + 0.1477\,[z(0.7983) - z(0.2017)]$	0.9294
10	$0.0192\,[z(0.99669) - z(0.00331)] + 0.0467\,[z(0.9830) - z(0.0170)]$ $+ 0.0776\,[z(0.9505) - z(0.0495)] + 0.1063\,[z(0.8876) - z(0.1124)]$ $+ 0.1228\,[z(0.7727) - z(0.2273)]$	0.9496
12	$0.0133\,[z(0.99776) - z(0.00224)] + 0.0323\,[z(0.9888) - z(0.0112)]$ $+ 0.0544\,[z(0.9680) - z(0.0320)] + 0.0767\,[z(0.9290) - z(0.0710)]$ $+ 0.0955\,[z(0.8628) - z(0.1372)] + 0.1041\,[z(0.7512) - z(0.2488)]$	0.9622
14	$0.00962\,[z(0.99843) - z(0.00157)] + 0.0235\,[z(0.99217) - z(0.00783)]$ $+ 0.0399\,[z(0.9779) - z(0.0221)] + 0.0571\,[z(0.9517) - z(0.0483)]$ $+ 0.0734\,[z(0.9086) - z(0.0914)] + 0.0860\,[z(0.8409) - z(0.1591)]$ $+ 0.0898\,[z(0.7332) - z(0.2668)]$	0.9706
16	$0.00725\,[z(0.99884) - z(0.00116)] + 0.0178\,[z(0.99425) - z(0.00575)]$ $+ 0.0305\,[z(0.9839) - z(0.0161)] + 0.0439\,[z(0.9652) - z(0.0348)]$ $+ 0.0572\,[z(0.9352) - z(0.0648)] + 0.0691\,[z(0.8894) - z(0.1106)]$ $+ 0.0776\,[z(0.8216) - z(0.1784)] + 0.0785\,[z(0.7178) - z(0.2822)]$	0.9764
18	$0.00510\,[z(0.999214) - z(0.000786)] + 0.0129\,[z(0.99606) - z(0.00394)]$ $+ 0.0225\,[z(0.9889) - z(0.0111)] + 0.0333\,[z(0.9759) - z(0.0241)]$ $+ 0.0448\,[z(0.9546) - z(0.0454)] + 0.0557\,[z(0.9226) - z(0.0774)]$ $+ 0.0652\,[z(0.8760) - z(0.1240)] + 0.0719\,[z(0.8090) - z(0.1910)]$ $+ 0.0716\,[z(0.7079) - z(0.2921)]$	0.9807
20	$0.00414\,[z(0.999367) - z(0.000633)] + 0.0103\,[z(0.99691) - z(0.00309)]$ $+ 0.0183\,[z(0.99128) - z(0.00872)] + 0.0273\,[z(0.9810) - z(0.0190)]$ $+ 0.0366\,[z(0.9647) - z(0.0353)] + 0.0456\,[z(0.9403) - z(0.0597)]$ $+ 0.0539\,[z(0.9060) - z(0.0940)] + 0.0608\,[z(0.8583) - z(0.1417)]$ $+ 0.0649\,[z(0.7919) - z(0.2081)] + 0.0632\,[z(0.6952) - z(0.3048)]$	0.9839

Table 3. Estimators of the mean of a normal distribution when $\sigma^2(\hat{\mu}) + \sigma^2(\hat{\sigma})$ is minimized

K	Estimators $(\hat{\mu})$	Efficiency $(\hat{\mu})$
6	0.0497 [z(0.0231) + z(0.9769)] + 0.1550 [z(0.1180) + z(0.8820)] + 0.2953 [z(0.3369) + z(0.6631)]	0.9459
8	0.0249 [z(0.0119) + z(0.9881)] + 0.0764 [z(0.0604) + z(0.9396)] + 0.1568 [z(0.1721) + z(0.8279)] + 0.2419 [z(0.3711) + z(0.6289)]	0.9659
10	0.0147 [z(0.00718) + z(0.99282)] + 0.0443 [z(0.0358) + z(0.9642)] + 0.0897 [z(0.1008) + z(0.8992)] + 0.1490 [z(0.2172) + z(0.7828)] + 0.2023 [z(0.3942) + z(0.6058)]	0.9767
12	0.0094 [z(0.00463) + z(0.99537)] + 0.0280 [z(0.0230) + z(0.9770)] + 0.0562 [z(0.0642) + z(0.9358)] + 0.0940 [z(0.1377) + z(0.8623)] + 0.1384 [z(0.2524) + z(0.7476)] + 0.1740 [z(0.4102) + z(0.5898)]	0.9830
14	0.0068 [z(0.00341) + z(0.99659)] + 0.0198 [z(0.0165) + z(0.9835)] + 0.0388 [z(0.0454) + z(0.9546)] + 0.0637 [z(0.0959) + z(0.9041)] + 0.0941 [z(0.1737) + z(0.8263)] + 0.1266 [z(0.2832) + z(0.7168)] + 0.1502 [z(0.4233) + z(0.5767)]	0.9873
16	0.0048 [z(0.00242) + z(0.99758)] + 0.0142 [z(0.0119) + z(0.9881)] + 0.0277 [z(0.0326) + z(0.9674)] + 0.0455 [z(0.0687) + z(0.9313)] + 0.0672 [z(0.1246) + z(0.8754)] + 0.0918 [z(0.2033) + z(0.7964)] + 0.1163 [z(0.3070) + z(0.6930)] + 0.1325 [z(0.4328) + z(0.5672)]	0.9900
18	0.0041 [z(0.00208) + z(0.99792)] + 0.0113 [z(0.00980) + z(0.99020)] + 0.0214 [z(0.0260) + z(0.9740)] + 0.0347 [z(0.0538) + z(0.9462)] + 0.0510 [z(0.0962) + z(0.9038)] + 0.0691 [z(0.1564) + z(0.8436)] + 0.0872 [z(0.2339) + z(0.7661)] + 0.1044 [z(0.3294) + z(0.6706)] + 0.1168 [z(0.4404) + z(0.5596)]	0.9922
20	0.0038 [z(0.00196) + z(0.99804)] + 0.0104 [z(0.00906) + z(0.99094)] + 0.0195 [z(0.0240)+ z(0.9760)] + 0.0304 [z(0.0488) + z(0.9512)] + 0.0426 [z(0.0853) + z(0.9147)] + 0.0558 [z(0.1341) + z(0.8659)] + 0.0694 [z(0.1967) + z(0.8033)] + 0.0818 [z(0.2722) + z(0.7278)] + 0.0909 [z(0.3589) + z(0.6411)] + 0.0954 [z(0.4522) + z(0.5478)]	0.9939

Table 4. Estimators of the standard deviation of a normal distribution when $\sigma^2(\hat{\mu}) + \sigma^2(\hat{\sigma})$ is minimized

K	Estimators $(\hat{\sigma})$	Efficiency $(\hat{\sigma})$
6	0.1088 [z(0.9769) −z(0.0231)] + 0.1952 [z(0.8820) −z(0.1180)] + 0.1228 [z(0.6631) −z(0.3369)]	0.8541
8	0.0600 [z(0.9881) −z(0.0119)] + 0.1249 [z(0.9396) −z(0.0604)] + 0.1528 [z(0.8279) −z(0.1721)] + 0.0789 [z(0.6289) −z(0.3711)]	0.9050
10	0.0379 [z(0.99282) −z(0.00718)] + 0.0829 [z(0.9642) −z(0.0358)] + 0.1181 [z(0.8992) −z(0.1008)] + 0.1184 [z(0.7828) −z(0.2172)] + 0.0540 [z(0.6058) −z(0.3942)]	0.9328
12	0.0255 [z(0.99537) −z(0.00463)] + 0.0576 [z(0.9770) −z(0.0230)] + 0.0877 [z(0.9358) −z(0.0642)] + 0.1047 [z(0.8623) −z(0.1377)] + 0.0933 [z(0.7476) −z(0.2524)] + 0.0394 [z(0.5898) −z(0.4102)]	0.9501
14	0.0191 [z(0.99659) −z(0.00341)] + 0.0432 [z(0.9835) −z(0.0165)] + 0.0671 [z(0.9546) −z(0.0454)] + 0.0849 [z(0.9041) −z(0.0959)] + 0.0898 [z(0.8263) −z(0.1737)] + 0.0731 [z(0.7168) −z(0.2832)] + 0.0291 [z(0.5767) −z(0.4233)]	0.9609
16	0.0141 [z(0.99758) −z(0.00242)] + 0.0327 [z(0.9881) −z(0.0119)] + 0.0521 [z(0.9674) −z(0.0326)] + 0.0688 [z(0.9313) −z(0.0687)] + 0.0787 [z(0.8754) −z(0.1246)] + 0.0770 [z(0.7964) −z(0.2033)] + 0.0590 [z(0.6930) −z(0.3070)] + 0.0225 [z(0.5672) −z(0.4328)]	0.9688
18	0.0121 [z(0.99792) −z(0.00208)] + 0.0270 [z(0.99020) −z(0.00980)] + 0.0424 [z(0.9740) −z(0.0260)] + 0.0567 [z(0.9462) −z(0.0538)] + 0.0674 [z(0.9038) −z(0.0962)] + 0.0709 [z(0.8436) −z(0.1564)] + 0.0640 [z(0.7661) −z(0.2339)] + 0.0464 [z(0.6706) −z(0.3294)] + 0.0174 [z(0.5596) −z(0.4404)]	0.9739
20	0.0113 [z(0.99804) −z(0.00196)] + 0.0251 [z(0.99094) −z(0.00906)] + 0.0392 [z(0.9760) −z(0.0240)] + 0.0512 [z(0.9512) −z(0.0488)] + 0.0594 [z(0.9147) −z(0.0853)] + 0.0627 [z(0.8659) −z(0.1341)] + 0.0601 [z(0.8033) −z(0.1967)] + 0.0503 [z(0.7278) −z(0.2722)] + 0.0334 [z(0.6411) −z(0.3589)] + 0.0116 [z(0.5478) −z(0.4522)]	0.9767

Table 5. Estimators of the mean of a normal distribution when $\sigma^2(\hat{\mu}) + 2\,\sigma^2(\hat{\sigma})$ is minimized

K	Estimators ($\hat{\mu}$)	Efficiency ($\hat{\mu}$)
6	0.0424 [z(0.0193) + z(0.9807)] + 0.1401 [z(0.1009) + z(0.8991)] + 0.3175 [z(0.3071) + z(0.6929)]	0.9368
8	0.0212 [z(0.00998) + z(0.99002)] + 0.0668 [z(0.0515) + z(0.9485)] + 0.1473 [z(0.1511) + z(0.8489)] + 0.2647) [z(0.3481) + z(0.6519)]	0.9610
10	0.0122 [z(0.00590) + z(0.99410)] + 0.0379 [z(0.0301) + z(0.9699)] + 0.0801 [z(0.0865) + z(0.9135)] + 0.1451 [z(0.1934) + z(0.8066)] + 0.2247 [z(0.3757) + z(0.6243)]	0.9736
12	0.0078 [z(0.00381) + z(0.99619)] + 0.0237 [z(0.0191) + z(0.9809)] + 0.0489 [z(0.0543) + z(0.9457)] + 0.0858 [z(0.1195) + z(0.8805)] + 0.1383 [z(0.2273) + z(0.7727)] + 0.1955 [z(0.3943) + z(0.6057)]	0.9808
14	0.0052 [z(0.00255) + z(0.99745)] + 0.0157 [z(0.0128) + z(0.9872)] + 0.0321 [z(0.0362) + z(0.9638)] + 0.0556 [z(0.0790) + z(0.9210)] + 0.0878 [z(0.1488) + z(0.8512)] + 0.1306 [z(0.2552) + z(0.7448)] + 0.1730 [z(0.4078) + z(0.5922)]	0.9853
16	0.0038 [z(0.00189) + z(0.99811)] + 0.0114 [z(0.00940) + z(0.99060)] + 0.0230 [z(0.0264) + z(0.9736)] + 0.0390 [z(0.0569) + z(0.9431)] + 0.0602 [z(0.1055) + z(0.8945)] + 0.0877 [z(0.1779) + z(0.8221)] + 0.1220 [z(0.2808) + z(0.7192)] + 0.1529 [z(0.4197) + z(0.5803)]	0.9887
18	0.0032 [z(0.00161) + z(0.99839)] + 0.0091 [z(0.00769) + z(0.99231)] + 0.0178 [z(0.0210) + z(0.9790)] + 0.0295 [z(0.0444) + z(0.9556)] + 0.0444 [z(0.0809) + z(0.9191)] + 0.0631 [z(0.1338) + z(0.8662)] + 0.0860 [z(0.2073) + z(0.7927)] + 0.1125 [z(0.3054) + z(0.6946)] + 0.1344 [z(0.4303) + z(0.5697)]	0.9912
20	0.0024 [z(0.00122) + z(0.99878)] + 0.0070 [z(0.00592) + z(0.99408)] + 0.0136 [z(0.0161) + z(0.9839)] + 0.0222 [z(0.0339) + z(0.9661)] + 0.0332 [z(0.0613) + z(0.9387)] + 0.0468 [z(0.1008) + z(0.8992)] + 0.0634 [z(0.1553) + z(0.8447)] + 0.0831 [z(0.2276) + z(0.7724)] + 0.1052 [z(0.3210) + z(0.6790)] + 0.1231 [z(0.4363) + z(0.5637)]	0.9927

Table 6. Estimators of the standard deviation of a normal distribution when $\sigma^2(\hat{\mu}) + 2\,\sigma^2(\hat{\sigma})$ is minimized

K	Estimators $(\hat{\sigma})$	Efficiency $(\hat{\sigma})$
6	0.0940 [z(0.9807) −z(0.0193)] + 0.1847 [z(0.8991) −z(0.1009)] + 0.1387 [z(0.6929) −z(0.3071)]	0.8649
8	0.0518 [z(0.99002) −z(0.00998)] + 0.1134 [z(0.9485) −z(0.0515)] + 0.1534 [z(0.8489) −z(0.1511)] + 0.0925 [z(0.6519) −z(0.3481)]	0.9107
10	0.0321 [z(0.99410) −z(0.00590)] + 0.0735 [z(0.9699) −z(0.0301)] + 0.1115 [z(0.9135) −z(0.0865)] + 0.1251 [z(0.8066) −z(0.1934)] + 0.0656 [z(0.6243) −z(0.3757)]	0.9369
12	0.0215 [z(0.99619) −z(0.00381)] + 0.0502 [z(0.9809) −z(0.0191)] + 0.0800 [z(0.9457) −z(0.0543)] + 0.1023 [z(0.8805) −z(0.1195)] + 0.1023 [z(0.7727) −z(0.2273)] + 0.0492 [z(0.6057) −z(0.3943)]	0.9531
14	0.0149 [z(0.99745) −z(0.00255)] + 0.0358 [z(0.9872) −z(0.0128)] + 0.0587 [z(0.9638) −z(0.0362)] + 0.0795 [z(0.9210) −z(0.0790)] + 0.0920 [z(0.8512) −z(0.1488)] + 0.0849 [z(0.7448) −z(0.2552)] + 0.0382 [z(0.5922) −z(0.4078)]	0.9639
16	0.0113 [z(0.99811) −z(0.00189)] + 0.0273 [z(0.99060) −z(0.00940)] + 0.0453 [z(0.9736) −z(0.0264)] + 0.0625 [z(0.9431) −z(0.0569)] + 0.0760 [z(0.8945) −z(0.1055)] + 0.0812 [z(0.8221) −z(0.1779)] + 0.0699 [z(0.7192) −z(0.2808)] + 0.0298 [z(0.5803) −z(0.4197)]	0.9711
18	0.0096 [z(0.99839) −z(0.00161)] + 0.0224 [z(0.99231) −z(0.00769)] + 0.0367 [z(0.9790) −z(0.0210)] + 0.0508 [z(0.9556) −z(0.0444)] + 0.0628 [z(0.9191) −z(0.0809)] + 0.0704 [z(0.8662) −z(0.1338)] + 0.0703 [z(0.7927) −z(0.2073)] + 0.0567 [z(0.6946) −z(0.3054)] + 0.0229 [z(0.5697) −z(0.4303)]	0.9760
20	0.0075 [z(0.99878) −z(0.00122)] + 0.0178 [z(0.99408) −z(0.00592)] + 0.0294 [z(0.9839) −z(0.0161)] + 0.0410 [z(0.9661) −z(0.0339)] + 0.0518 [z(0.9387) −z(0.0613)] + 0.0603 [z(0.8992) −z(0.1008)] + 0.0646 [z(0.8447) −z(0.1553)] + 0.0621 [z(0.7724) −z(0.2276)] + 0.0484 [z(0.6790) −z(0.3210)] + 0.0192 [z(0.5637) −z(0.4363)]	0.9801

Table 7. Estimators of the mean of a normal distribution when $\sigma^2(\hat{\mu}) + 3\,\sigma^2(\hat{\sigma})$ is minimized

K	Estimators $(\hat{\mu})$	Efficiency $(\hat{\mu})$
6	0.0389 [z(0.0175) + z(0.9825)] + 0.1306 [z(0.0922) + z(0.9078)] + 0.3305 [z(0.2858) + z(0.7142)]	0.9277
8	0.0196 [z(0.00921) + z(0.99079)] + 0.0626 [z(0.0476) + z(0.9524)] + 0.1404 [z(0.1407) + z(0.8593)] + 0.2774 [z(0.3314) + z(0.6686)]	0.9568
10	0.0114 [z(0.00548) + z(0.99452)] + 0.0351 [z(0.0277) + z(0.9723)] + 0.0750 [z(0.0801) + z(0.9199)] + 0.1405 [z(0.1811) + z(0.8189)] + 0.2380 [z(0.3615) + z(0.6385)]	0.9710
12	0.0071 [z(0.00346) + z(0.99654)] + 0.0218 [z(0.0175) + z(0.9825)] + 0.0455 [z(0.0500) + z(0.9500)] + 0.0812 [z(0.1108) + z(0.8892)] + 0.1360 [z(0.2140) + z(0.7860)] + 0.2084 [z(0.3823) + z(0.6177)]	0.9790
14	0.0048 [z(0.00238) + z(0.99762)] + 0.0146 [z(0.0119) + z(0.9881)] + 0.0300 [z(0.0337) + z(0.9663)] + 0.0522 [z(0.0738) + z(0.9262)] + 0.0838 [z(0.1396) + z(0.8604)] + 0.1299 [z(0.2421) + z(0.7579)] + 0.1847 [z(0.3983) + z(0.6017)]	0.9842
16	0.0036 [z(0.00177) + z(0.99823)] + 0.0106 [z(0.00876) + z(0.99124)] + 0.0215 [z(0.0245) + z(0.9755)] + 0.0365 [z(0.0530) + z(0.9470)] + 0.0569 [z(0.0987) + z(0.9013)] + 0.0846 [z(0.1676) + z(0.8324)] + 0.1227 [z(0.2682) + z(0.7318)] + 0.1636 [z(0.4116) + z(0.5884)]	0.9897
18	0.0025 [z(0.0012) + z(0.99873)] + 0.0077 [z(0.00633) + z(0.99367)] + 0.0155 [z(0.0177) + z(0.9823)] + 0.0263 [z(0.0383) + z(0.9617)] + 0.0405 [z(0.0712) + z(0.9288)] + 0.0592 [z(0.1201) + z(0.8799)] + 0.0838 [z(0.1900) + z(0.8100)] + 0.1161 [z(0.2878) + z(0.7122)] + 0.1484 [z(0.4209) + z(0.5791)]	0.9902
20	0.0022 [z(0.00112) + z(0.99888)] + 0.0064 [z(0.00541) + z(0.99459)] + 0.0126 [z(0.0149) + z(0.9851)] + 0.0208 [z(0.0314) + z(0.9686)] + 0.0314 [z(0.0572) + z(0.9428)] + 0.0445 [z(0.0947) + z(0.9053)] + 0.0609 [z(0.1465) + z(0.8535)] + 0.0816 [z(0.2166) + z(0.7834)] + 0.1077 [z(0.3094) + z(0.6906)] + 0.1319 [z(0.4307) + z(0.5693)]	0.9922

Table 8. Estimators of the standard deviation of a normal distribution when $\sigma^2(\hat{\mu}) + 3\,\sigma^2(\hat{\sigma})$ is minimized

K	Estimators ($\hat{\sigma}$)	Efficiency ($\hat{\sigma}$)
6	0.0865 [z(0.9825) −z(0.0175)] + 0.1764 [z(0.9078) −z(0.0922)] + 0.1478 [z(0.7142) −z(0.2858)]	0.8714
8	0.0484 [z(0.99079) −z(0.00921)] + 0.1075 [z(0.9524) −z(0.0476)] + 0.1512 [z(0.8593) −z(0.1407)] + 0.1004 [z(0.6686) −z(0.3314)]	0.9139
10	0.0300 [z(0.99452) −z(0.00548)] + 0.0690 [z(0.9723) −z(0.0277)] + 0.1072 [z(0.9199) −z(0.0801)] + 0.1264 [z(0.8189) −z(0.1811)] + 0.0728 [z(0.6385) −z(0.3615)]	0.9389
12	0.0198 [z(0.99654) −z(0.00346)] + 0.0469 [z(0.9825) −z(0.0175)] + 0.0761 [z(0.9500) −z(0.0500)] + 0.1000 [z(0.8892) −z(0.1108)] + 0.1058 [z(0.7860) −z(0.2140)] + 0.0553 [z(0.6177) −z(0.3823)]	0.9545
14	0.0140 [z(0.99762) −z(0.00238)] + 0.0337 [z(0.9881) −z(0.0119)] + 0.0557 [z(0.9663) −z(0.0337)] + 0.0765 [z(0.9262) −z(0.0738)] + 0.0909 [z(0.8604) −z(0.1396)] + 0.0888 [z(0.7579) −z(0.2421)] + 0.0432 [z(0.6017) −z(0.3983)]	0.9649
16	0.0107 [z(0.99823) −z(0.00177)] + 0.0256 [z(0.99124) −z(0.00876)] + 0.0428 [z(0.9755) −z(0.0245)] + 0.0598 [z(0.9470) −z(0.0530)] + 0.0739 [z(0.9013) −z(0.0987)] + 0.0815 [z(0.8324) −z(0.1676)] + 0.0742 [z(0.7318) −z(0.2682)] + 0.0338 [z(0.5884) −z(0.4116)]	0.9719
18	0.0079 [z(0.99873) −z(0.00127)] + 0.0194 [z(0.99367) −z(0.00633)] + 0.0329 [z(0.9823) −z(0.0177)] + 0.0470 [z(0.9617) −z(0.0383)] + 0.0600 [z(0.9288) −z(0.0712)] + 0.0698 [z(0.8799) −z(0.1201)] + 0.0734 [z(0.8100) −z(0.1900)] + 0.0636 [z(0.7122) −z(0.2878)] + 0.0278 [z(0.5791) −z(0.4209)]	0.9772
20	0.0069 [z(0.99888) −z(0.00112)] + 0.0166 [z(0.99459) −z(0.00541)] + 0.0277 [z(0.9851) −z(0.0149)] + 0.0391 [z(0.9686) −z(0.0314)] + 0.0499 [z(0.9428) −z(0.0572)] + 0.0588 [z(0.9053) −z(0.0947)] + 0.0643 [z(0.8535) −z(0.1465)] + 0.0638 [z(0.7834) −z(0.2166)] + 0.0525 [z(0.6906) −z(0.3094)] + 0.0219 [z(0.5693) −z(0.4307)]	0.9807

Table 9. Estimators of the mean of a normal distribution with minimum variance, for $p_1 \geqq 0.01$

K	Estimators ($\hat{\mu}$)	Efficiency ($\hat{\mu}$)
6	0.0968 [z(0.0540) + z(0.9460)] + 0.1787 [z(0.1915) + z(0.8085)] + 0.2245 [z(0.3898) + z(0.6102)]	0.9560
8	0.0559 [z(0.0310) + z(0.9690)] + 0.1119 [z(0.1154) + z(0.8846)] + 0.1550 [z(0.2481) + z(0.7519)] + 0.1772 [z(0.4216) + z(0.5874)]	0.9722
10	0.0366 [z(0.0203) + z(0.9797)] + 0.0751 [z(0.0768) + z(0.9232)] + 0.1086 [z(0.1684) + z(0.8316)] + 0.1334 [z(0.2887) + z(0.7113)] + 0.1463 [z(0.4274) + z(0.5726)]	0.9808
12	0.0246 [z(0.0135) + z(0.9865)] + 0.0522 [z(0.0525) + z(0.9475)] + 0.0786 [z(0.1178) + z(0.8822)] + 0.1012 [z(0.2075) + z(0.7925)] + 0.1174 [z(0.3163) + z(0.6837)] + 0.1260 [z(0.4373) + z(0.5627)]	0.9859
14	0.0177 [z(0.01) + z(0.99)] + 0.0365 [z(0.0370) + z(0.9630)] + 0.0571 [z(0.0839) + z(0.9161)] + 0.0764 [z(0.1507) + z(0.8493)] + 0.0934 [z(0.2349) + z(0.7651)] + 0.1063 [z(0.3349) + z(0.6651)] + 0.1126 [z(0.4438) + z(0.5562)]	0.9892
16	0.0170 [z(0.01) + z(0.99)] + 0.0320 [z(0.0342) + z(0.9658)] + 0.0482 [z(0.0746) + z(0.9254)] + 0.0627 [z(0.1299) + z(0.8701)] + 0.0754 [z(0.1988) + z(0.8012)] + 0.0842 [z(0.2790) + z(0.7210)] + 0.0891 [z(0.3648) + z(0.6352)] + 0.0914 [z(0.4549) + z(0.5451)]	0.9914
18	0.0162 [z(0.01) + z(0.99)] + 0.0274 [z(0.0306) + z(0.9694)] + 0.0423 [z(0.0661) + z(0.9339)] + 0.0541 [z(0.1148) + z(0.8852)] + 0.0640 [z(0.1730) + z(0.8270)] + 0.0718 [z(0.2417) + z(0.7583)] + 0.0747 [z(0.3148) + z(0.6852)] + 0.0754 [z(0.3893) + z(0.6107)] + 0.0741 [z(0.4641) + z(0.5359)]	0.9928
20	0.0138 [z(0.01) + z(0.99)] + 0.0174 [z(0.0223) + z(0.9777)] + 0.0310 [z(0.0473) + z(0.9527)] + 0.0427 [z(0.0848) + z(0.9152)] + 0.0531 [z(0.1323) + z(0.8677)] + 0.0627 [z(0.1904) + z(0.8096)] + 0.0673 [z(0.2565) + z(0.7435)] + 0.0687 [z(0.3234) + z(0.6766)] + 0.0704 [z(0.3925) + z(0.6075)] + 0.0729 [z(0.4629) + z(0.5371)]	0.9938

Table 10. Estimators of the standard deviation of a normal distribution with minimum variance, for $p_1 \geqq 0.01$

K	Estimators $(\hat{\sigma})$	Efficiency $(\hat{\sigma})$
6	0.0549 [z(0.9896) − z(0.0104)] + 0.1244 [z(0.9452) − z(0.0548)] + 0.1825 [z(0.8304) − z(0.1696)]	0.8943
8	0.0456 [z(0.99) − z(0.01)] + 0.0771 [z(0.9624) − z(0.0376)] + 0.1139 [z(0.9026) − z(0.0974)] + 0.1376 [z(0.7863) − z(0.2137)]	0.9260
10	0.0410 [z(0.99) − z(0.01)] + 0.0543 [z(0.9703) − z(0.0297)] + 0.0783 [z(0.9324) − z(0.0676)] + 0.0991 [z(0.8668) − z(0.1332)] + 0.1092 [z(0.7548) − z(0.2452)]	0.9408
12	0.0382 [z(0.99) − z(0.01)] + 0.0419 [z(0.9746) − z(0.0254)] + 0.0585 [z(0.9474) − z(0.0526)] + 0.0739 [z(0.9037) − z(0.0963)] + 0.0858 [z(0.8361) − z(0.1639)] + 0.0890 [z(0.7293) − z(0.2707)]	0.9488
14	0.0363 [z(0.99) − z(0.01)] + 0.0345 [z(0.9771) − z(0.0229)] + 0.0467 [z(0.9560) − z(0.0440)] + 0.0582 [z(0.9235) − z(0.0765)] + 0.0678 [z(0.8766) − z(0.1234)] + 0.0742 [z(0.8090) − z(0.1910)] + 0.0737 [z(0.7082) − z(0.2918)]	0.9536
16	0.0348 [z(0.99) − z(0.01)] + 0.0288 [z(0.9792) − z(0.0208)] + 0.0381 [z(0.9622) − z(0.0378)] + 0.0471 [z(0.9373) − z(0.0627)] + 0.0554 [z(0.9023) − z(0.0977)] + 0.0618 [z(0.8540) − z(0.1460)] + 0.0652 [z(0.7878) − z(0.2122)] + 0.0630 [z(0.6920) − z(0.3080)]	0.9567
18	0.0339 [z(0.99) − z(0.01)] + 0.0255 [z(0.9804) − z(0.0196)] + 0.0333 [z(0.9657) − z(0.0343)] + 0.0406 [z(0.9446) − z(0.0554)] + 0.0471 [z(0.9162) − z(0.0838)] + 0.0525 [z(0.8784) − z(0.1216)] + 0.0558 [z(0.8287) − z(0.1713)] + 0.0562 [z(0.7637) − z(0.2363)] + 0.0522 [z(0.6741) − z(0.3259)]	0.9588
20	0.0332 [z(0.99) − z(0.01)] + 0.0231 [z(0.9812) − z(0.0188)] + 0.0296 [z(0.9682) − z(0.0318)] + 0.0360 [z(0.9501) − z(0.0499)] + 0.0418 [z(0.9256) − z(0.0744)] + 0.0461 [z(0.8940) − z(0.1060)] + 0.0491 [z(0.8541) − z(0.1459)] + 0.0502 [z(0.8037) − z(0.1963)] + 0.0485 [z(0.7398) − z(0.2602)] + 0.0425 [z(0.6548) − z(0.3452)]	0.9603

Table 11. Estimators of the mean of a normal distribution when $\sigma^2(\hat{\mu}) + \sigma^2(\hat{\sigma})$ is minimized, for $p_1 \geqq 0.01$

K	Estimators ($\hat{\mu}$)	Efficiency ($\hat{\mu}$)
6	0.0497 [z(0.0231) + z(0.9769)] + 0.1550 [z(0.1180) + z(0.8820)] + 0.2953 [z(0.3369) + z(0.6631)]	0.9459
8	0.0249 [z(0.0119) + z(0.9881)] + 0.0764 [z(0.0604) + z(0.9396)] + 0.1568 [z(0.1721) + z(0.8279)] + 0.2419 [z(0.3711) + z(0.6289)]	0.9659
10	0.0189 [z(0.01) + z(0.99)] + 0.0469 [z(0.0414) + z(0.9586)] + 0.0905 [z(0.1080) + z(0.8920)] + 0.1467 [z(0.2239) + z(0.7761)] + 0.1970 [z(0.3971) + z(0.6029)]	0.9777
12	0.0169 [z(0.01) + z(0.99)] + 0.0327 [z(0.0330) + z(0.9670)] + 0.0589 [z(0.0780) + z(0.9220)] + 0.0933 [z(0.1524) + z(0.8476)] + 0.1333 [z(0.2643) + z(0.7357)] + 0.1644 [z(0.4150) + z(0.5850)]	0.9842
14	0.0155 [z(0.01) + z(0.99)] + 0.0243 [z(0.0280) + z(0.9720)] + 0.0413 [z(0.0601) + z(0.9399)] + 0.0638 [z(0.1118) + z(0.8882)] + 0.0913 [z(0.1882) + z(0.8118)] + 0.1208 [z(0.2936) + z(0.7064)] + 0.1430 [z(0.4265) + z(0.5735)]	0.9880
16	0.0148 [z(0.01) + z(0.99)] + 0.0200 [z(0.0252) + z(0.9748)] + 0.0321 [z(0.0509) + z(0.9491)] + 0.0475 [z(0.0901) + z(0.9099)] + 0.0666 [z(0.1465) + z(0.8535)] + 0.0879 [z(0.2232) + z(0.7768)] + 0.1085 [z(0.3212) + z(0.6788)] + 0.1226 [z(0.4376) + z(0.5624)]	0.9906
18	0.0138 [z(0.01) + z(0.99)] + 0.0155 [z(0.0222) + z(0.9778)] + 0.0242 [z(0.0418) + z(0.9582)] + 0.0354 [z(0.0711) + z(0.9289)] + 0.0496 [z(0.1131) + z(0.8869)] + 0.0660 [z(0.1705) + z(0.8295)] + 0.0834 [z(0.2447) + z(0.7553)] + 0.1005 [z(0.3362) + z(0.6638)] + 0.1116 [z(0.4436) + z(0.5564)]	0.9921
20	0.0131 [z(0.01) + z(0.99)] + 0.0126 [z(0.0201) + z(0.9799)] + 0.0189 [z(0.0356) + z(0.9644)] + 0.0274 [z(0.0584) + z(0.9416)] + 0.0384 [z(0.0910) + z(0.9090)] + 0.0516 [z(0.1357) + z(0.8643)] + 0.0656 [z(0.1942) + z(0.8058)] + 0.0799 [z(0.2663) + z(0.7337)] + 0.0926 [z(0.3530) + z(0.6470)] + 0.0999 [z(0.4496) + z(0.5504)]	0.9932

Table 12. Estimators of the standard deviation of a normal distribution when $\sigma^2(\hat{\mu}) + \sigma^2(\hat{\sigma})$ is minimized, for $p_1 \geqq 0.01$

K	Estimators $(\hat{\sigma})$	Efficiency $(\hat{\sigma})$
6	0.1088 [z(0.9769) −z(0.0231)] + 0.1952 [z(0.8820) −z(0.1180)] + 0.1228 [z(0.6631) −z(0.3369)]	0.8541
8	0.0600 [z(0.9881) −z(0.0119)] + 0.1249 [z(0.9396) −z(0.0604)] + 0.1528 [z(0.8279) −z(0.1721)] + 0.0789 [z(0.6289) −z(0.3711)]	0.9050
10	0.0470 [z(0.99) −z(0.01)] + 0.0849 [z(0.9586) −z(0.0414)] + 0.1160 [z(0.8920) −z(0.1080)] + 0.1139 [z(0.7761) −z(0.2239)] + 0.0514 [z(0.6029) −z(0.3971)]	0.9294
12	0.0427 [z(0.99) −z(0.01)] + 0.0625 [z(0.9670) −z(0.0330)] + 0.0867 [z(0.9220) −z(0.0780)] + 0.0986 [z(0.8476) −z(0.1524)] + 0.0857 [z(0.7359) −z(0.2643)] + 0.0357 [z(0.5850) −z(0.4150)]	0.9417
14	0.0396 [z(0.99) −z(0.01)] + 0.0483 [z(0.9720) −z(0.0280)] + 0.0664 [z(0.9399) −z(0.0601)] + 0.0802 [z(0.8882) −z(0.1118)] + 0.0830 [z(0.8118) −z(0.1882)] + 0.0669 [z(0.7064) −z(0.2936)] + 0.0266 [z(0.5735) −z(0.4265)]	0.9489
16	0.0378 [z(0.99) −z(0.01)] + 0.0405 [z(0.9748) −z(0.0252)] + 0.0544 [z(0.9491) −z(0.0509)] + 0.0658 [z(0.9099) −z(0.0901)] + 0.0723 [z(0.8533) −z(0.1565)] + 0.0689 [z(0.7768) −z(0.2232)] + 0.0516 [z(0.6788) −z(0.3212)] + 0.0196 [z(0.5624) −z(0.4376)]	0.9532
18	0.0358 [z(0.99) −z(0.01)] + 0.0324 [z(0.9778) −z(0.0222)] + 0.0432 [z(0.9582) −z(0.0418)] + 0.0536 [z(0.9289) −z(0.0711)] + 0.0619 [z(0.8869) −z(0.1131)] + 0.0648 [z(0.8295) −z(0.1705)] + 0.0592 [z(0.7553) −z(0.2447)] + 0.0434 [z(0.6638) −z(0.3362)] + 0.0162 [z(0.5564) −z(0.4436)]	0.9564
20	0.0342 [z(0.99) −z(0.01)] + 0.0267 [z(0.9799) −z(0.0201)] + 0.0353 [z(0.9644) −z(0.0356)] + 0.0444 [z(0.9416) −z(0.0584)] + 0.0529 [z(0.9090) −z(0.0910)] + 0.0585 [z(0.8643) −z(0.1357)] + 0.0583 [z(0.8058) −z(0.1942)] + 0.0512 [z(0.7337) −z(0.2663)] + 0.0359 [z(0.6470) −z(0.3530)] + 0.0130 [z(0.5504) −z(0.4496)]	0.9585

Table 13. Estimators of the mean of a normal distribution when $\sigma^2(\hat{\mu}) + 2\,\sigma^2(\hat{\sigma})$ is minimized, for $p_1 \geqq 0.01$

K	Estimators $(\hat{\mu})$	Efficiency $(\hat{\mu})$
6	0.0424 [z(0.0193) + z(0.9807)] + 0.1401 [z(0.1009) + z(0.8991)] + 0.3175 [z(0.3071) + z(0.6929)]	0.9368
8	0.0212 [z(0.00998) + z(0.99002)] + 0.0668 [z(0.0515) + z(0.9485)] + 0.1473 [z(0.1511) + z(0.8489)] + 0.2647 [z(0.3481) + z(0.6519)]	0.9610
10	0.0183 [z(0.01) + z(0.99)] + 0.0423 [z(0.0384) + z(0.9616)] + 0.0825 [z(0.0982) + z(0.9018)] + 0.1428 [z(0.2058) + z(0.7942)] + 0.2141 [z(0.3823) + z(0.6177)]	0.9758
12	0.0164 [z(0.01) + z(0.99)] + 0.0296 [z(0.0310) + z(0.9690)] + 0.0534 [z(0.0714) + z(0.9286)] + 0.0870 [z(0.1394) + z(0.8606)] + 0.1329 [z(0.2460) + z(0.7540)] + 0.1807 [z(0.4031) + z(0.5969)]	0.9830
14	0.0152 [z(0.01) + z(0.99)] + 0.0226 [z(0.0268) + z(0.9732)] + 0.0382 [z(0.0566) + z(0.9434)] + 0.0592 [z(0.1042) + z(0.8958)] + 0.0870 [z(0.1758) + z(0.8242)] + 0.1222 [z(0.2779) + z(0.7221)] + 0.1556 [z(0.4179) + z(0.5821)]	0.9873
16	0.0142 [z(0.01) + z(0.99)] + 0.0176 [z(0.0234) + z(0.9766)] + 0.0282 [z(0.0460) + z(0.9540)] + 0.0422 [z(0.0805) + z(0.9195)] + 0.0605 [z(0.1309) + z(0.8691)] + 0.0843 [z(0.2017) + z(0.7983)] + 0.1138 [z(0.2994) + z(0.7006)] + 0.1392 [z(0.4275) + z(0.5725)]	0.9897
18	0.0135 [z(0.01) + z(0.99)] + 0.0143 [z(0.0211) + z(0.9789)] + 0.0222 [z(0.0393) + z(0.9607)] + 0.0323 [z(0.0659) + z(0.9341)] + 0.0454 [z(0.1044) + z(0.8956)] + 0.0616 [z(0.1570) + z(0.8430)] + 0.0819 [z(0.2277) + z(0.7723)] + 0.1050 [z(0.3202) + z(0.6798)] + 0.1238 [z(0.4359) + z(0.5641)]	0.9916
20	0.0130 [z(0.01) + z(0.99)] + 0.0119 [z(0.0196) + z(0.9804)] + 0.0176 [z(0.0342) + z(0.9658)] + 0.0253 [z(0.0553) + z(0.9447)] + 0.0351 [z(0.0851) + z(0.9149)] + 0.0477 [z(0.1259) + z(0.8741)] + 0.0628 [z(0.1807) + z(0.8193)] + 0.0791 [z(0.2513) + z(0.7487)] + 0.0968 [z(0.3381) + z(0.6619)] + 0.1107 [z(0.4434) + z(0.5566)]	0.9929

Table 14. Estimators of the standard deviation of a normal distribution when $\sigma^2(\hat{\mu}) + 2\,\sigma^2(\hat{\sigma})$ is minimized, for $p_1 \geqq 0.01$

K	Estimators $(\hat{\sigma})$	Efficiency $(\hat{\sigma})$
6	0.0940 [z(0.9807) −z(0.0193)] + 0.1847 [z(0.8991) −z(0.1009)] + 0.1387 [z(0.6929) −z(0.3071)]	0.8649
8	0.0518 [z(0.99002) −z(0.00998)] + 0.1134 [z(0.9485) −z(0.0515)] + 0.1534 [z(0.8489) −z(0.1511)] + 0.0925 [z(0.6519) −z(0.3481)]	0.9107
10	0.0456 [z(0.99) −z(0.01)] + 0.0776 [z(0.9616) −z(0.0384)] + 0.1097 [z(0.9018) −z(0.0982)] + 0.1179 [z(0.7942) −z(0.2058)] + 0.0599 [z(0.6177) −z(0.3823)]	0.9319
12	0.0416 [z(0.99) −z(0.01)] + 0.0573 [z(0.9690) −z(0.0310)] + 0.0808 [z(0.9286) −z(0.0714)] + 0.0965 [z(0.8606) −z(0.1394)] + 0.0918 [z(0.7540) −z(0.2460)] + 0.0424 [z(0.5969) −z(0.4031)]	0.9432
14	0.0389 [z(0.99) −z(0.01)] + 0.0453 [z(0.9732) −z(0.0268)] + 0.0626 [z(0.9434) −z(0.0566)] + 0.0768 [z(0.8958) −z(0.1042)] + 0.0829 [z(0.8242) −z(0.1758)] + 0.0724 [z(0.7221) −z(0.2779)] + 0.0314 [z(0.5821) −z(0.4179)]	0.9498
16	0.0367 [z(0.99) −z(0.01)] + 0.0361 [z(0.9766) −z(0.0234)] + 0.0491 [z(0.9540) −z(0.0460)] + 0.0610 [z(0.9195) −z(0.0805)] + 0.0698 [z(0.8691) −z(0.1309)] + 0.0719 [z(0.7983) −z(0.2017)] + 0.0603 [z(0.7006) −z(0.2994)] + 0.0251 [z(0.5725) −z(0.4275)]	0.9541
18	0.0350 [z(0.99) −z(0.01)] + 0.0300 [z(0.9789) −z(0.0211)] + 0.0404 [z(0.9607) −z(0.0393)] + 0.0502 [z(0.9341) −z(0.0659)] + 0.0588 [z(0.8956) −z(0.1044)] + 0.0637 [z(0.8430) −z(0.1570)] + 0.0624 [z(0.7723) −z(0.2277)] + 0.0495 [z(0.6798) −z(0.3202)] + 0.0198 [z(0.5641) −z(0.4359)]	0.9569
20	0.0339 [z(0.99) −z(0.01)] + 0.0254 [z(0.9804) −z(0.0196)] + 0.0333 [z(0.9658) −z(0.0342)] + 0.0416 [z(0.9447) −z(0.0553)] + 0.0497 [z(0.9149) −z(0.0851)] + 0.0562 [z(0.8741) −z(0.1259)] + 0.0588 [z(0.8193) −z(0.1807)] + 0.0544 [z(0.7487) −z(0.2513)] + 0.0409 [z(0.6619) −z(0.3381)] + 0.0158 [z(0.5566) −z(0.4434)]	0.9589

Table 15. Estimators of the mean of a normal distribution when $\sigma^2(\hat{\mu}) + 3\,\sigma^2(\hat{\sigma})$ is minimized, for $p_1 \geqq 0.01$

K	Estimators $(\hat{\mu})$	Efficiency $(\hat{\mu})$
6	0.0389 [z(0.0175) + z(0.9825)] + 0.1306 [z(0.0922) + z(0.9078)] + 0.3305 [z(0.2858) + z(0.7142)]	0.9277
8	0.0208 [z(0.01) + z(0.99)] + 0.0632 [z(0.0492) + z(0.9508)] + 0.1406 [z(0.1426) + z(0.8574)] + 0.2754 [z(0.3327) + z(0.6673)]	0.9574
10	0.0179 [z(0.01) + z(0.99)] + 0.0398 [z(0.0368) + z(0.9632)] + 0.0778 [z(0.0929) + z(0.9071)] + 0.1386 [z(0.1947) + z(0.8053)] + 0.2259 [z(0.3700) + z(0.6300)]	0.9739
12	0.0162 [z(0.01) + z(0.99)] + 0.0281 [z(0.0300) + z(0.9700)] + 0.0504 [z(0.0682) + z(0.9318)] + 0.0828 [z(0.1323) + z(0.8677)] + 0.1311 [z(0.2345) + z(0.7655)] + 0.1914 [z(0.3935) + z(0.6065)]	0.9819
14	0.0150 [z(0.01) + z(0.99)] + 0.0212 [z(0.0257) + z(0.9743)] + 0.0358 [z(0.0536) + z(0.9464)] + 0.0560 [z(0.0985) + z(0.9015)] + 0.0838 [z(0.1664) + z(0.8336)] + 0.1227 [z(0.2662) + z(0.7338)] + 0.1655 [z(0.4103) + z(0.5897)]	0.9865
16	0.0141 [z(0.01) + z(0.99)] + 0.0168 [z(0.0229) + z(0.9771)] + 0.0266 [z(0.0442) + z(0.9558)] + 0.0400 [z(0.0768) + z(0.9232)] + 0.0578 [z(0.1248) + z(0.8752)] + 0.0817 [z(0.1928) + z(0.8072)] + 0.1144 [z(0.2882) + z(0.7118)] + 0.1486 [z(0.4202) + z(0.5798)]	0.9892
18	0.0133 [z(0.01) + z(0.99)] + 0.0134 [z(0.0206) + z(0.9794)] + 0.0206 [z(0.0373) + z(0.9627)] + 0.0302 [z(0.0623) + z(0.9377)] + 0.0426 [z(0.0981) + z(0.9019)] + 0.0584 [z(0.1479) + z(0.8521)] + 0.0793 [z(0.2149) + z(0.7851)] + 0.1071 [z(0.3063) + z(0.6937)] + 0.1345 [z(0.4291) + z(0.5709)]	0.9910
20	0.0130 [z(0.01) + z(0.99)] + 0.0120 [z(0.0196) + z(0.9804)] + 0.0180 [z(0.0345) + z(0.9655)] + 0.0254 [z(0.0560) + z(0.9440)] + 0.0346 [z(0.0855) + z(0.9145)] + 0.0461 [z(0.1254) + z(0.8746)] + 0.0600 [z(0.1779) + z(0.8221)] + 0.0772 [z(0.2452) + z(0.7548)] + 0.0981 [z(0.3319) + z(0.6681)] + 0.1156 [z(0.4400) + z(0.5600)]	0.9927

Table 16. Estimators of the standard deviation of a normal distribution when $\sigma^2(\hat{\mu}) + 3\,\sigma^2(\hat{\sigma})$ is minimized, for $p_1 \geqq 0.01$

K	Estimators ($\hat{\sigma}$)	Efficiency ($\hat{\sigma}$)
6	0.0865 [z(0.9825) −z(0.0175)] + 0.1764 [z(0.9078) −z(0.0922)] + 0.1478 [z(0.7142) −z(0.2858)]	0.8714
8	0.0509 [z(0.99) −z(0.01)] + 0.1077 [z(0.9508) −z(0.0492)] + 0.1502 [z(0.8574) −z(0.1426)] + 0.0992 [z(0.6673) −z(0.3327)]	0.9134
10	0.0448 [z(0.99) −z(0.01)] + 0.0736 [z(0.9632) −z(0.0368)] + 0.1056 [z(0.9071) −z(0.0929)] + 0.1189 [z(0.8053) −z(0.1947)] + 0.0660 [z(0.6300) −z(0.3700)]	0.9332
12	0.0410 [z(0.99) −z(0.01)] + 0.0547 [z(0.9700) −z(0.0300)] + 0.0774 [z(0.9318) −z(0.0682)] + 0.0943 [z(0.8677) −z(0.1323)] + 0.0945 [z(0.7655) −z(0.2345)] + 0.0472 [z(0.6065) −z(0.3935)]	0.9440
14	0.0383 [z(0.99) −z(0.01)] + 0.0427 [z(0.9743) −z(0.0257)] + 0.0596 [z(0.9464) −z(0.0536)] + 0.0744 [z(0.9015) −z(0.0985)] + 0.0828 [z(0.8336) −z(0.1664)] + 0.0764 [z(0.7338) −z(0.2662)] + 0.0353 [z(0.5897) −z(0.4103)]	0.9504
16	0.0363 [z(0.99) −z(0.01)] + 0.0346 [z(0.9771) −z(0.0229)] + 0.0468 [z(0.9558) −z(0.0442)] + 0.0588 [z(0.9232) −z(0.0768)] + 0.0684 [z(0.8752) −z(0.1248)] + 0.0722 [z(0.8072) −z(0.1928)] + 0.0638 [z(0.7118) −z(0.2882)] + 0.0284 [z(0.5798) −z(0.4202)]	0.9545
18	0.0346 [z(0.99) −z(0.01)] + 0.0283 [z(0.9794) −z(0.0206)] + 0.0379 [z(0.9627) −z(0.0373)] + 0.0478 [z(0.9377) −z(0.0623)] + 0.0567 [z(0.9019) −z(0.0981)] + 0.0627 [z(0.8521) −z(0.1479)] + 0.0637 [z(0.7851) −z(0.2149)] + 0.0544 [z(0.6937) −z(0.3063)] + 0.0233 [z(0.5709) −z(0.4291)]	0.9573
20	0.0339 [z(0.99) −z(0.01)] + 0.0256 [z(0.9804) −z(0.0196)] + 0.0338 [z(0.9655) −z(0.0345)] + 0.0417 [z(0.9440) −z(0.0560)] + 0.0488 [z(0.9145) −z(0.0855)] + 0.0546 [z(0.8746) −z(0.1254)] + 0.0569 [z(0.8221) −z(0.1779)] + 0.0543 [z(0.7548) −z(0.2452)] + 0.0429 [z(0.6681) −z(0.3319)] + 0.0172 [z(0.5600) −z(0.4400)]	0.9590

Table 17. Estimators of the mean of a normal distribution with minimum variance, for $p_1 \geqq 0.025$

K	Estimators ($\hat{\mu}$)	Efficiency ($\hat{\mu}$)
6	0.0968 [z(0.0540) + z(0.9460)] + 0.1787 [z(0.1915) + z(0.8085)] + 0.2245 [z(0.3898) + z(0.6102)]	0.9560
8	0.0559 [z(0.0310) + z(0.9690)] + 0.1119 [z(0.1154) + z(0.8846)] + 0.1550 [z(0.2481) + z(0.7519)] + 0.1772 [z(0.4126) + z(0.5874)]	0.9722
10	0.0426 [z(0.025) + z(0.975)] + 0.0768 [z(0.0839) + z(0.9161)] + 0.1084 [z(0.1764) + z(0.8236)] + 0.1304 [z(0.2949) + z(0.7051)] + 0.1418 [z(0.4297) + z(0.5703)]	0.9806
12	0.0387 [z(0.025) + z(0.975)] + 0.0566 [z(0.0698) + z(0.9302)] + 0.0786 [z(0.1371) + z(0.8629)] + 0.0972 [z(0.2243) + z(0.7757)] + 0.1109 [z(0.3278) + z(0.6722)] + 0.1180 [z(0.4415) + z(0.5585)]	0.9851
14	0.0352 [z(0.025) + z(0.975)] + 0.0414 [z(0.0583) + z(0.9417)] + 0.0581 [z(0.1077) + z(0.8923)] + 0.0740 [z(0.1735) + z(0.8265)] + 0.0882 [z(0.2537) + z(0.7463)] + 0.0987 [z(0.3473) + z(0.6527)] + 0.1044 [z(0.4476) + z(0.5524)]	0.9876
16	0.0331 [z(0.025) + z(0.975)] + 0.0327 [z(0.0517) + z(0.9483)] + 0.0455 [z(0.0905) + z(0.9095)] + 0.0585 [z(0.1422) + z(0.8578)] + 0.0709 [z(0.2064) + z(0.7936)] + 0.0807 [z(0.2825) + z(0.7175)] + 0.0870 [z(0.3656) + z(0.6344)] + 0.0916 [z(0.4539) + z(0.5461)]	0.9892
18	0.0310 [z(0.025) + z(0.975)] + 0.0257 [z(0.0458) + z(0.9542)] + 0.0374 [z(0.0770) + z(0.9230)] + 0.0494 [z(0.1210) + z(0.8790)] + 0.0591 [z(0.1749) + z(0.8251)] + 0.0673 [z(0.2380) + z(0.7620)] + 0.0731 [z(0.3080) + z(0.6920)] + 0.0780 [z(0.3822) + z(0.6178)] + 0.0790 [z(0.4620) + z(0.5380)]	0.9903
20	0.0312 [z(0.025) + z(0.975)] + 0.0259 [z(0.0465) + z(0.9535)] + 0.0359 [z(0.0769) + z(0.9231)] + 0.0465 [z(0.1184) + z(0.8816)] + 0.0555 [z(0.1691) + z(0.8309)] + 0.0624 [z(0.2285) + z(0.7715)] + 0.0613 [z(0.2924) + z(0.7076)] + 0.0614 [z(0.3490) + z(0.6510)] + 0.0624 [z(0.4140) + z(0.5860)] + 0.0575 [z0.4721) + z(0.5279)]	0.9910

Table 18. Estimators of the standard deviation of a normal distribution with minimum variance, for $p_1 \geqq 0.025$

K	Estimators $(\hat{\sigma})$	Efficiency $(\hat{\sigma})$
6	$0.0953\,[z(0.975)-z(0.025)] + 0.1249\,[z(0.9203)-z(0.0797)] + 0.1609\,[z(0.8036)-z(0.1964)]$	0.8777
8	$0.0842\,[z(0.975)-z(0.025)] + 0.0825\,[z(0.9389)-z(0.0611)] + 0.1070\,[z(0.8741)-z(0.1259)] + 0.1198\,[z(0.7609)-z(0.2391)]$	0.8965
10	$0.0779\,[z(0.975)-z(0.025)] + 0.0606\,[z(0.9483)-z(0.0517)] + 0.0772\,[z(0.9052)-z(0.0948)] + 0.0903\,[z(0.8380)-z(0.1620)] + 0.0945\,[z(0.7316)-z(0.2684)]$	0.9051
12	$0.0741\,[z(0.975)-z(0.025)] + 0.0484\,[z(0.9535)-z(0.0465)] + 0.0600\,[z(0.9213)-z(0.0787)] + 0.0701\,[z(0.8744)-z(0.1256)] + 0.0767\,[z(0.8073)-z(0.1927)] + 0.0764\,[z(0.7069)-z(0.2931)]$	0.9098
14	$0.0713\,[z(0.975)-z(0.025)] + 0.0402\,[z(0.9571)-z(0.0429)] + 0.0490\,[z(0.9314)-z(0.0686)] + 0.0565\,[z(0.8960)-z(0.1040)] + 0.0626\,[z(0.8481)-z(0.1519)] + 0.0660\,[z(0.7826)-z(0.2174)] + 0.0636\,[z(0.6882)-z(0.3118)]$	0.9126
16	$0.0693\,[z(0.975)-z(0.025)] + 0.0346\,[z(0.9596)-z(0.0404)] + 0.0417\,[z(0.9381)-z(0.0619)] + 0.0478\,[z(0.9095)-z(0.0905)] + 0.0529\,[z(0.8719)-z(0.1281)] + 0.0563\,[z(0.8232)-z(0.1768)] + 0.0569\,[z(0.7589)-z(0.2411)] + 0.0526\,[z(0.6705)-z(0.3295)]$	0.9144
18	$0.0679\,[z(0.975)-z(0.025)] + 0.0303\,[z(0.9614)-z(0.0386)] + 0.0359\,[z(0.9432)-z(0.0568)] + 0.0412\,[z(0.9194)-z(0.0806)] + 0.0461\,[z(0.8888)-z(0.1112)] + 0.0501\,[z(0.8492)-z(0.1508)] + 0.0514\,[z(0.7990)-z(0.2010)] + 0.0493\,[z(0.7354)-z(0.2646)] + 0.0428\,[z(0.6517)-z(0.3483)]$	0.9156
20	$0.0676\,[z(0.975)-z(0.025)] + 0.0294\,[z(0.9617)-z(0.0383)] + 0.0348\,[z(0.9442)-z(0.0558)] + 0.0393\,[z(0.9210)-z(0.0790)] + 0.0417\,[z(0.8932)-z(0.1068)] + 0.0439\,[z(0.8592)-z(0.1408)] + 0.0451\,[z(0.8180)-z(0.1820)] + 0.0439\,[z(0.7672)-z(0.2328)] + 0.0396\,[z(0.7068)-z(0.2932)] + 0.0326\,[z(0.6311)-z(0.3689)]$	0.9164

Table 19. Estimators of the mean of a normal distribution when $\sigma^2(\hat{\mu}) + \sigma^2(\hat{\sigma})$ is minimized, for $p_1 \geqq 0.025$

K	Estimators $(\hat{\mu})$	Efficiency $(\hat{\mu})$
6	0.0525 [z(0.025) + z(0.975)] + 0.1554 [z(0.1211) + z(0.8789)] + 0.2921 [z(0.3389) + z(0.6611)]	0.9470
8	0.0429 [z(0.025) + z(0.975)] + 0.0848 [z(0.0834) + z(0.9166)] + 0.1527 [z(0.1978) + z(0.8022)] + 0.2196 [z(0.3841) + z(0.6159)]	0.9698
10	0.0379 [z(0.025) + z(0.975)] + 0.0562 [z(0.0663) + z(0.9337)] + 0.0932 [z(0.1393) + z(0.8607)] + 0.1377 [z(0.2522) + z(0.7478)] + 0.1750 [z(0.4097) + z(0.5903)]	0.9793
12	0.0347 [z(0.025) + z(0.975)] + 0.0405 [z(0.0561) + z(0.9439)] + 0.0636 [z(0.1070) + z(0.8930)] + 0.0921 [z(0.1836) + z(0.8164)] + 0.1230 [z(0.2897) + z(0.7103)] + 0.1461 [z(0.4256) + z(0.5744)]	0.9841
14	0.0325 [z(0.025) + z(0.975)] + 0.0313 [z(0.0499) + z(0.9501)] + 0.0463 [z(0.0881) + z(0.9119)] + 0.0649 [z(0.1426) + z(0.8574)] + 0.0874 [z(0.2175) + z(0.7825)] + 0.1107 [z(0.3160) + z(0.6840)] + 0.1269 [z(0.4355) + z(0.5645)]	0.9869
16	0.0308 [z(0.025) + z(0.975)] + 0.0246 [z(0.0450) + z(0.9550)] + 0.0352 [z(0.0745) + z(0.9255)] + 0.0485 [z(0.1157) + z(0.8843)] + 0.0644 [z(0.1712) + z(0.8288)] + 0.0825 [z(0.2439) + z(0.7561)] + 0.1002 [z(0.3347) + z(0.6653)] + 0.1138 [z(0.4415) + z(0.5585)]	0.9886
18	0.0296 [z(0.025) + z(0.975)] + 0.0199 [z(0.0418 + z(0.9582)] + 0.0274 [z(0.0649) + z(0.9351)] + 0.0381 [z(0.0968) + z(0.9032)] + 0.0509 [z(0.1414) + z(0.8586)] + 0.0637 [z(0.1981) + z(0.8019)] + 0.0786 [z(0.2679) + z(0.7321)] + 0.0923 [z(0.3539) + z(0.6461)] + 0.0995 [z(0.4501) + z(0.5499)]	0.9898
20	0.0288 [z(0.025) + z(0.975)] + 0.0172 [z(0.0398) + z(0.9602)] + 0.0229 [z(0.0593) + z(0.9407)] + 0.0303 [z(0.0857) + z(0.9143)] + 0.0387 [z(0.1200) + z(0.8800)] + 0.0493 [z(0.1627) + z(0.8373)] + 0.0620 [z(0.2184) + z(0.7816)] + 0.0737 [z(0.2859) + z(0.7141)] + 0.0849 [z(0.3643) + z(0.6357)] + 0.0922 [z(0.4537) + z(0.5463)]	0.9906

Table 20. Estimators of the standard deviation of a normal distribution when $\sigma^2(\hat{\mu}) + \sigma^2(\hat{\sigma})$ is minimized, for $p_1 \geqq 0.025$

K	Estimators $(\hat{\sigma})$	Efficiency $(\hat{\sigma})$
6	$0.1139\,[z(0.975) - z(0.025)] + 0.1938\,[z(0.8789) - z(0.1211)] + 0.1204\,[z(0.6611) - z(0.3389)]$	0.8524
8	$0.0962\,[z(0.975) - z(0.025)] + 0.1266\,[z(0.9166) - z(0.0834)] + 0.1376\,[z(0.8022) - z(0.1978)] + 0.0664\,[z(0.6159) - z(0.3841)]$	0.8843
10	$0.0868\,[z(0.975) - z(0.025)] + 0.0914\,[z(0.9337) - z(0.0663)] + 0.1087\,[z(0.8607) - z(0.1393)] + 0.0975\,[z(0.7478) - z(0.2522)] + 0.0418\,[z(0.5903) - z(0.4097)]$	0.8982
12	$0.0805\,[z(0.975) - z(0.025)] + 0.0695\,[z(0.9439) - z(0.0561)] + 0.0852\,[z(0.8930) - z(0.1070)] + 0.0891\,[z(0.8164) - z(0.1836)] + 0.0725\,[z(0.7103) - z(0.2897)] + 0.0290\,[z(0.5744) - z(0.4256)]$	0.9055
14	$0.0763\,[z(0.975) - z(0.025)] + 0.0556\,[z(0.950) - z(0.0499)] + 0.0676\,[z(0.9119) - z(0.0881)] + 0.0747\,[z(0.8574) - z(0.1426)] + 0.0730\,[z(0.7825) - z(0.2175)] + 0.0564\,[z(0.6840) - z(0.3160)] + 0.0218\,[z(0.5645) - z(0.4355)]$	0.9098
16	$0.0729\,[z(0.975) - z(0.025)] + 0.0450\,[z(0.9550) - z(0.0450)] + 0.0549\,[z(0.9255) - z(0.0745)] + 0.0625\,[z(0.8843) - z(0.1157)] + 0.0657\,[z(0.8288) - z(0.1712)] + 0.0613\,[z(0.7561) - z(0.2439)] + 0.0456\,[z(0.6653) - z(0.3347)] + 0.0174\,[z(0.5585) - z(0.4415)]$	0.9125
18	$0.0704\,[z(0.975) - z(0.025)] + 0.0372\,[z(0.9582) - z(0.0418)] + 0.0448\,[z(0.9351) - z(0.0649)] + 0.0533\,[z(0.9032) - z(0.0968)] + 0.0589\,[z(0.8586) - z(0.1414)] + 0.0582\,[z(0.8019) - z(0.1981)] + 0.0520\,[z(0.7321) - z(0.2679)] + 0.0371\,[z(0.6461) - z(0.3539)] + 0.0134\,[z(0.5499) - z(0.4501)]$	0.9143
20	$0.0688\,[z(0.975) - z(0.025)] + 0.0326\,[z(0.9602) - z(0.0398)] + 0.0385\,[z(0.9407) - z(0.0593)] + 0.0448\,[z(0.9143) - z(0.0857)] + 0.0491\,[z(0.8800) - z(0.1200)] + 0.0521\,[z(0.8373) - z(0.1627)] + 0.0519\,[z(0.7816) - z(0.2184)] + 0.0447\,[z(0.7141) - z(0.2859)] + 0.0313\,[z(0.6357) - z(0.3643)] + 0.0115\,[z(0.5463) - z(0.4537)]$	0.9155

Table 21. Estimators of the mean of a normal distribution when $\sigma^2(\hat{\mu}) + 2\,\sigma^2(\hat{\sigma})$ is minimized, for $p_1 \geqq 0.025$

K	Estimators $(\hat{\mu})$	Efficiency $(\hat{\mu})$
6	0.0507 [z(0.025) + z(0.975)] + 0.1424 [z(0.1110) + z(0.8890)] + 0.3069 [z(0.3154) + z(0.6846)]	0.9414
8	0.0416 [z(0.025) + z(0.975)] + 0.0772 [z(0.0780) + z(0.9220)] + 0.1455 [z(0.1826) + z(0.8174)] + 0.2357 [z(0.3678) + z(0.6322)]	0.9675
10	0.0369 [z(0.025) + z(0.975)] + 0.0510 [z(0.0626) + z(0.9374)] + 0.0860 [z(0.1287) + z(0.8713)] + 0.1356 [z(0.2350) + z(0.7650)] + 0.1905 [z(0.3971) + z(0.6029)]	0.9781
12	0.0339 [z(0.025) + z(0.975)] + 0.0370 [z(0.0535) + z(0.9465)] + 0.0580 [z(0.1000) + z(0.9000)] + 0.0862 [z(0.1696) + z(0.8304)] + 0.1243 [z(0.2720) + z(0.7280)] + 0.1606 [z(0.4155) + z(0.5845)]	0.9833
14	0.0320 [z(0.025) + z(0.975)] + 0.0290 [z(0.0482) + z(0.9518)] + 0.0430 [z(0.0836) + z(0.9164)] + 0.0608 [z(0.1344) + z(0.8656)] + 0.0838 [z(0.2048) + z(0.7952)] + 0.1127 [z(0.3010) + z(0.6990)] + 0.1387 [z(0.4276) + z(0.5724)]	0.9863
16	0.0303 [z(0.025) + z(0.975)] + 0.0224 [z(0.0435) + z(0.9565)] + 0.0319 [z(0.0701) + z(0.9299)] + 0.0441 [z(0.1074) + z(0.8926)] + 0.0596 [z(0.1582) + z(0.8418)] + 0.0801 [z(0.2264) + z(0.7736)] + 0.1050 [z(0.3175) + z(0.6825)] + 0.1266 [z(0.4341) + z(0.5659)]	0.9880
18	0.0292 [z(0.025) + z(0.975)] + 0.0192 [z(0.0410) + z(0.9590)] + 0.0266 [z(0.0636) + z(0.9364)] + 0.0354 [z(0.0942) + z(0.9058)] + 0.0466 [z(0.1341) + z(0.8659)] + 0.0608 [z(0.1873) + z(0.8127)] + 0.0773 [z(0.2551) + z(0.7449)] + 0.0954 [z(0.3407) + z(0.6593)] + 0.1095 [z(0.4438) + z(0.5562)]	0.9895
20	0.0287 [z(0.025) + z(0.975)] + 0.0173 [z(0.0396) + z(0.9604)] + 0.0232 [z(0.0597) + z(0.9403)] + 0.0298 [z(0.0859) + z(0.9141)] + 0.0380 [z(0.1192) + z(0.8808)] + 0.0485 [z(0.1618) + z(0.8382)] + 0.0608 [z(0.2159) + z(0.7841)] + 0.0740 [z(0.2826) + z(0.7174)] + 0.0861 [z(0.3626) + z(0.6374)] + 0.0936 [z(0.4527) + z(0.5473)]	0.9905

Table 22. Estimators of the standard deviation of a normal distribution when $\sigma^2(\hat{\mu}) + 2\,\sigma^2(\hat{\sigma})$ is minimized, for $p_1 \geqq 0.025$

K	Estimators $(\hat{\sigma})$	Efficiency $(\hat{\sigma})$
6	0.1098 [z(0.975) −z(0.025)] + 0.1819 [z(0.8890) −z(0.1110)] + 0.1304 [z(0.6846) −z(0.3154)]	0.8588
8	0.0935 [z(0.975) −z(0.025)] + 0.1174 [z(0.9220) −z(0.0780)] + 0.1376 [z(0.8174) −z(0.1826)] + 0.0754 [z(0.6322) −z(0.3678)]	0.8869
10	0.0847 [z(0.975) −z(0.025)] + 0.0843 [z(0.9374) −z(0.0626)] + 0.1042 [z(0.8713) −z(0.1287)] + 0.1024 [z(0.7650) −z(0.2350)] + 0.0490 [z(0.6029) −z(0.3971)]	0.8997
12	0.0789 [z(0.975) −z(0.025)] + 0.0643 [z(0.9465) −z(0.0535)] + 0.0800 [z(0.9000) −z(0.1000)] + 0.0879 [z(0.8304) −z(0.1696)] + 0.0789 [z(0.7280) −z(0.2720)] + 0.0348 [z(0.5845) −z(0.4155)]	0.9065
14	0.0752 [z(0.975) −z(0.025)] + 0.0521 [z(0.9518) −z(0.0482)] + 0.0640 [z(0.9164) −z(0.0836)] + 0.0722 [z(0.8656) −z(0.1344)] + 0.0737 [z(0.7952) −z(0.2048)] + 0.0617 [z(0.6990) −z(0.3010)] + 0.0259 [z(0.5724) −z(0.4276)]	0.9104
16	0.0717 [z(0.975) −z(0.025)] + 0.0414 [z(0.9565) −z(0.0435)] + 0.0507 [z(0.9299) −z(0.0701)] + 0.0589 [z(0.8926) −z(0.1074)] + 0.0641 [z(0.8418) −z(0.1582)] + 0.0640 [z(0.7736) −z(0.2264)] + 0.0523 [z(0.6825) −z(0.3175)] + 0.0215 [z(0.5659) −z(0.4341)]	0.9130
18	0.0698 [z(0.975) −z(0.025)] + 0.0360 [z(0.9590) −z(0.0410)] + 0.0438 [z(0.9364) −z(0.0636)] + 0.0502 [z(0.9058) −z(0.0942)] + 0.0554 [z(0.8659) −z(0.1341)] + 0.0579 [z(0.8127) −z(0.1873)] + 0.0543 [z(0.7449) −z(0.2551)] + 0.0414 [z(0.6593) −z(0.3407)] + 0.0161 [z(0.5562) −z(0.4438)]	0.9145
20	0.0687 [z(0.975) −z(0.025)] + 0.0327 [z(0.9604) −z(0.0396)] + 0.0390 [z(0.9403) −z(0.0597)] + 0.0440 [z(0.9141) −z(0.0859)] + 0.0483 [z(0.8808) −z(0.1192)] + 0.0515 [z(0.8382) −z(0.1618)] + 0.0513 [z(0.7841) −z(0.2159)] + 0.0455 [z(0.7174) −z(0.2826)] + 0.0323 [z(0.6374) −z(0.3626)] + 0.0118 [z(0.5473) −z(0.4527)]	0.9156

Table 23. Estimators of the mean of a normal distribution when $\sigma^2(\hat{\mu}) + 3\,\sigma^2(\hat{\sigma})$ is minimized, for $p_1 \geqq 0.025$

K	Estimators $(\hat{\mu})$	Efficiency $(\hat{\mu})$
6	0.0498 [z(0.025) + z(0.975)] + 0.1340 [z(0.1054) + z(0.8946)] + 0.3162 [z(0.2977) + z(0.7023)]	0.9353
8	0.0409 [z(0.025) + z(0.975)] + 0.0730 [z(0.0751) + z(0.9249)] + 0.1398 [z(0.1739) + z(0.8261)] + 0.2463 [z(0.3544) + z(0.6456)]	0.9652
10	0.0363 [z(0.025) + z(0.975)] + 0.0482 [z(0.0606) + z(0.9394)] + 0.0815 [z(0.1228) + z(0.8772)] + 0.1327 [z(0.2241) + z(0.7759)] + 0.2013 [z(0.3864) + z(0.6136)]	0.9768
12	0.0335 [z(0.025) + z(0.975)] + 0.0353 [z(0.0524) + z(0.9476)] + 0.0551 [z(0.0963) + z(0.9037)] + 0.0826 [z(0.1628) + z(0.8372)] + 0.1230 [z(0.2612) + z(0.7388)] + 0.1705 [z(0.4068) + z(0.5932)]	0.9825
14	0.0315 [z(0.025) + z(0.975)] + 0.0273 [z(0.0469) + z(0.9531)] + 0.0401 [z(0.0799) + z(0.9201)] + 0.0573 [z(0.1272) + z(0.8728)] + 0.0810 [z(0.1944) + z(0.8056)] + 0.1137 [z(0.2886) + z(0.7114)] + 0.1491 [z(0.4198) + z(0.5802)]	0.9858
16	0.0299 [z(0.025) + z(0.975)] + 0.0214 [z(0.0426) + z(0.9574)] + 0.0301 [z(0.0679) + z(0.9321)] + 0.0417 [z(0.1028) + z(0.8972)] + 0.0575 [z(0.1514) + z(0.8486)] + 0.0785 [z(0.2175) + z(0.7825)] + 0.1065 [z(0.3079) + z(0.6921)] + 0.1344 [z(0.4284) + z(0.5716)]	0.9876
18	0.0294 [z(0.025) + z(0.975)] + 0.0194 [z(0.0415) + z(0.9585)] + 0.0265 [z(0.0639) + z(0.9361)] + 0.0353 [z(0.0945) + z(0.9055)] + 0.0459 [z(0.1343) + z(0.8657)] + 0.0594 [z(0.1859) + z(0.8141)] + 0.0754 [z(0.2525) + z(0.7475)] + 0.0952 [z(0.3355) + z(0.6645)] + 0.1135 [z(0.4411) + z(0.5589)]	0.9894
20	0.0284 [z(0.025) + z(0.975)] + 0.0165 [z(0.0389) + z(0.9611)] + 0.0218 [z(0.0580) + z(0.9420)] + 0.0286 [z(0.0824) + z(0.9176)] + 0.0376 [z(0.1154) + z(0.8846)] + 0.0479 [z(0.1574) + z(0.8426)] + 0.0600 [z(0.2107) + z(0.7893)] + 0.0734 [z(0.2767) + z(0.7233)] + 0.0870 [z(0.3564) + z(0.6436)] + 0.0988 [z(0.4487) + z(0.5513)]	0.9904

Table 24. Estimators of the standard deviation of a normal distribution when $\sigma^2(\hat{\mu}) + 3\,\sigma^2(\hat{\sigma})$ is minimized, for $p_1 \geqq 0.025$

K	Estimators $(\hat{\sigma})$	Efficiency $(\hat{\sigma})$
6	0.1073 [z(0.975) −z(0.025)] + 0.1736 [z(0.8946) −z(0.1054)] + 0.1365 [z(0.7023) −z(0.2977)]	0.8629
8	0.0921 [z(0.975) −z(0.025)] + 0.1122 [z(0.9249) −z(0.0751)] + 0.1360 [z(0.8261) −z(0.1739)] + 0.0814 [z(0.6456) −z(0.3544)]	0.8885
10	0.0835 [z(0.975) −z(0.025)] + 0.0823 [z(0.9394) −z(0.0606)] + 0.1009 [z(0.8772) −z(0.1228)] + 0.1043 [z(0.7759) −z(0.2241)] + 0.0541 [z(0.6136) −z(0.3864)]	0.9005
12	0.0782 [z(0.975) −z(0.025)] + 0.0617 [z(0.9476) −z(0.0524)] + 0.0770 [z(0.9037) −z(0.0963)] + 0.0865 [z(0.8372) −z(0.1628)] + 0.0816 [z(0.7388) −z(0.2612)] + 0.0388 [z(0.5932) −z(0.4068)]	0.9069
14	0.0743 [z(0.975) −z(0.025)] + 0.0492 [z(0.9531) −z(0.0469)] + 0.0607 [z(0.9201) −z(0.0799)] + 0.0700 [z(0.8728) −z(0.1272)] + 0.0742 [z(0.8056) −z(0.1944)] + 0.0660 [z(0.7114) −z(0.2886)] + 0.0297 [z(0.5802) −z(0.4198)]	0.9107
16	0.0711 [z(0.975) −z(0.025)] + 0.0396 [z(0.9574) −z(0.0426)] + 0.0484 [z(0.9321) −z(0.0679)] + 0.0567 [z(0.8972) −z(0.1028)] + 0.0634 [z(0.8486) −z(0.1514)] + 0.0651 [z(0.7825) −z(0.2175)] + 0.0557 [z(0.6921) −z(0.3079)] + 0.0241 [z(0.5716) −z(0.4284)]	0.9132
18	0.0701 [z(0.975) −z(0.025)] + 0.0364 [z(0.9585) −z(0.0415)] + 0.0435 [z(0.9361) −z(0.0639)] + 0.0500 [z(0.9055) −z(0.0945)] + 0.0546 [z(0.8657) −z(0.1343)] + 0.0569 [z(0.8141) −z(0.1859)] + 0.0537 [z(0.7475) −z(0.2525)] + 0.0423 [z(0.6645) −z(0.3355)] + 0.0173 [z(0.5589) −z(0.4411)]	0.9146
20	0.0681 [z(0.975) −z(0.025)] + 0.0313 [z(0.9611) −z(0.0389)] + 0.0370 [z(0.9420) −z(0.0580)] + 0.0428 [z(0.9176) −z(0.0824)] + 0.0486 [z(0.8846) −z(0.1154)] + 0.0517 [z(0.8426) −z(0.1574)] + 0.0518 [z(0.7893) −z(0.2107)] + 0.0465 [z(0.7233) −z(0.2767)] + 0.0340 [z(0.6436) −z(0.3564)] + 0.0131 [z(0.5513) −z(0.4487)]	0.9157

Table 25. Efficiencies of the estimates of the mean and standard deviation of a normal distribution from quantiles, with no restriction on p_1

K	Min. $\sigma^2(\hat{\mu})$	Min. $\sigma^2(\hat{\sigma})$	Min. $[\sigma^2(\hat{\mu}) + \sigma^2(\hat{\sigma})]$		Min. $[\sigma^2(\hat{\mu}) + 2\sigma^2(\hat{\sigma})]$		Min. $[\sigma^2(\hat{\mu}) + 3\sigma^2(\hat{\sigma})]$	
	Eff $(\hat{\mu})$	Eff $(\hat{\sigma})$	Eff $(\hat{\mu})$	Eff $(\hat{\sigma})$	Eff $(\hat{\mu})$	Eff $(\hat{\sigma})$	Eff $(\hat{\mu})$	Eff $(\hat{\sigma})$
6	0.9560	0.8943	0.9459	0.8541	0.9368	0.8649	0.9277	0.8714
8	0.9722	0.9294	0.9659	0.9050	0.9610	0.9107	0.9568	0.9139
10	0.9808	0.9496	0.9767	0.9328	0.9736	0.9369	0.9710	0.9389
12	0.9859	0.9622	0.9830	0.9501	0.9808	0.9531	0.9790	0.9545
14	0.9892	0.9706	0.9873	0.9609	0.9853	0.9639	0.9842	0.9649
16	0.9915	0.9764	0.9900	0.9688	0.9887	0.9711	0.9878	0.9719
18	0.9931	0.9807	0.9922	0.9739	0.9912	0.9760	0.9902	0.9772
20	0.9943	0.9839	0.9939	0.9767	0.9927	0.9801	0.9922	0.9807

Table 26. Efficiencies of the estimates of the mean and standard deviation of a normal distribution from quantiles, for $p_1 \geqq 0.01$

K	Min. $\sigma^2(\hat{\mu})$	Min. $\sigma^2(\hat{\sigma})$	Min. $[\sigma^2(\hat{\mu}) + \sigma^2(\hat{\sigma})]$		Min. $[\sigma^2(\hat{\mu}) + 2\sigma^2(\hat{\sigma})]$		Min. $[\sigma^2(\hat{\mu}) + 3\sigma^2(\hat{\mu})]$	
	Eff $(\hat{\mu})$	Eff $(\hat{\sigma})$	Eff $(\hat{\mu})$	Eff $(\hat{\sigma})$	Eff $(\hat{\mu})$	Eff $(\hat{\sigma})$	Eff $(\hat{\mu})$	Eff $(\hat{\sigma})$
6	0.9560	0.8943	0.9459	0.8541	0.9368	0.8649	0.9277	0.8714
8	0.9722	0.9260	0.9659	0.9050	0.9610	0.9107	0.9574	0.9134
10	0.9808	0.9408	0.9777	0.9294	0.9758	0.9319	0.9739	0.9332
12	0.9859	0.9488	0.9842	0.9417	0.9830	0.9432	0.9819	0.9440
14	0.9892	0.9536	0.9880	0.9489	0.9873	0.9498	0.9865	0.9504
16	0.9914	0.9567	0.9906	0.9532	0.9897	0.9541	0.9892	0.9545
18	0.9928	0.9588	0.9921	0.9564	0.9917	0.9569	0.9910	0.9573
20	0.9938	0.9603	0.9932	0.9585	0.9929	0.9589	0.9927	0.9590

Table 27. Efficiencies of the estimates of the mean and standard deviation of a normal distribution from quantiles, for $p_1 \geqq 0.025$

K	Min. $\sigma^2(\hat{\mu})$ Eff $(\hat{\mu})$	Min. $\sigma^2(\hat{\sigma})$ Eff $(\hat{\sigma})$	Min. $[\sigma^2(\hat{\mu}) + \sigma^2(\hat{\sigma})]$		Min. $[\sigma^2(\hat{\sigma}) + 2\sigma^2(\hat{\sigma})]$		Min. $[\sigma^2(\hat{\sigma}) + 3\sigma^2(\hat{\sigma})]$	
			Eff $(\hat{\mu})$	Eff $(\hat{\sigma})$	Eff $(\hat{\mu})$	Eff $(\hat{\sigma})$	Eff $(\hat{\mu})$	Eff $(\hat{\sigma})$
6	0.9560	0.8777	0.9470	0.8524	0.9414	0.8585	0.9353	0.8629
8	0.9722	0.8965	0.9698	0.8843	0.9675	0.8869	0.9652	0.8885
10	0.9806	0.9051	0.9793	0.8982	0.9781	0.8997	0.9768	0.9005
12	0.9851	0.9098	0.9841	0.9055	0.9833	0.9065	0.9825	0.9069
14	0.9876	0.9126	0.9869	0.9098	0.9863	0.9104	0.9858	0.9107
16	0.9892	0.9144	0.9886	0.9125	0.9880	0.9130	0.9876	0.9132
18	0.9903	0.9156	0.9898	0.9143	0.9895	0.9145	0.9894	0.9146
20	0.9910	0.9164	0.9906	0.9155	0.9905	0.9156	0.9904	0.9157

Table 28. Rejection criteria for a normally distributed hypothesized population

n	σ_{y_1}	σ_{y_2}	$\varepsilon = 0.05$		$\varepsilon = 0.01$	
			K_1	K_2	K_1	K_2
200	0.1859	0.2252	0.416	−0.360	0.552	−0.220
250	0.1663	0.2014	0.372	−0.406	0.467	−0.281
300	0.1518	0.1839	0.339	−0.440	0.427	−0.326
400	0.1315	0.1592	0.294	−0.489	0.370	−0.390
500	0.1177	0.1424	0.263	−0.522	0.331	−0.433
600	0.1074	0.1300	0.240	−0.546	0.302	−0.465
700	0.0994	0.1204	0.222	−0.565	0.279	−0.490
800	0.0930	0.1126	0.208	−0.580	0.261	−0.510
900	0.0877	0.1062	0.196	−0.592	0.246	−0.527
1000	0.0832	0.1007	0.186	−0.603	0.234	−0.541

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